Lecture-09: Limit Theorems

1 Growth of renewal counting processes

Lemma 1.1. *Consider the counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *associated with* i.i.d. *inter-renewal time sequence* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *with finite mean* $\mathbb{E}X_n < \infty$. *Let* $N_\infty \triangleq \lim_{t\to\infty} N_t$, *then* $P\{N_\infty = \infty\} = 1$.

Proof. It suffices to show $P\{N_\infty < \infty\} = 0$. Since $\mathbb{E}[X_n] < \infty$, we have $P\{X_n = \infty\} = 0$ and

$$P\{N_\infty < \infty\} = P\bigcup_{n\in\mathbb{N}}\{N_\infty) < n\} = P\bigcup_{n\in\mathbb{N}}\{S_n = \infty\} = P\bigcup_{n\in\mathbb{N}}\{X_n = \infty\} \leqslant \sum_{n\in\mathbb{N}} P\{X_n = \infty\} = 0. \quad (1)$$

□

Corollary 1.2. *For delayed renewal processes with finite mean of first renewal instant and subsequent inter-renewal times,* $P\left\{\lim_{t\to\infty} N_t^D = \infty\right\} = 1$.

We observed that the number of renewals N_t increases to infinity with the length of the duration t. We will show that the growth of N_t is asymptotically linear with time t, and we will find this coefficient of linear growth of N_t with time t.

1.1 Strong law for renewal processes

Theorem 1.3 (Strong law). *For a renewal counting process with inter-arrival times having a finite mean, we have*

$$\lim_{t\to\infty}\frac{N_t}{t} = \frac{1}{\mu} \text{ almost surely.} \quad (2)$$

Proof. Note that S_{N_t} represents the time of last renewal before t, and S_{N_t+1} represents the time of first renewal after time t. Clearly, we have $S_{N_t} \leqslant t < S_{N_t+1}$. Dividing by N_t, we get

$$\frac{S_{N_t}}{N_t} \leqslant \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t}. \quad (3)$$

Since N_t increases monotonically to infinity as t grows large, we can apply strong law of large numbers to the sum $S_{N_t} = \sum_{i=1}^{N_t} X_i$, to get $\lim_{t\to\infty}\frac{S_{N_t}}{N_t} = \mu$ almost surely. Hence the result follows. □

Corollary 1.4. *For a delayed renewal process with finite inter-arrival durations,* $\lim_{t\to\infty}\frac{N_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.5. Suppose, you are in a casino with infinitely many games. Every game has a probability of win X, *i.i.d.* uniformly distributed between $(0,1)$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins. Let $N(n)$ denote the number of losses in n plays. Then the fraction of wins $P_W(n)$ is given by

$$P_W(n) = \frac{n - N(n)}{n}.$$

We pick a strategy where any game is selected to play, and continue to be played till the first loss. Note that, time till first loss is geometrically distributed with mean $\frac{1}{1-X}$. We shall show that this

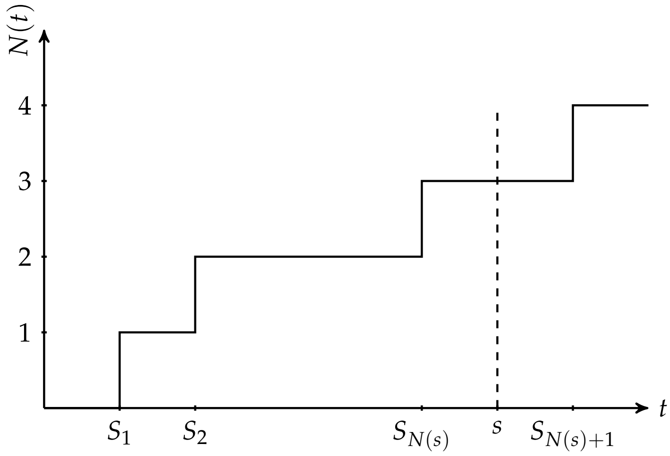

Figure 1: Time of last renewal

fraction approaches unity as $n \to \infty$. By the previous proposition, we have:

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}[\text{Time till first loss}]}$$
$$= \frac{1}{\mathbb{E}\left[\frac{1}{1-X}\right]} = \frac{1}{\infty} = 0$$

Hence Renewal theorems can be used to compute these long term averages. We'll have many such theorems in the following sections.

1.2 Elementary renewal theorem

Basic renewal theorem implies $\frac{N_t}{t}$ converges to $\frac{1}{\mu}$ almost surely. We are next interested in convergence of the ratio $\frac{m_t}{t}$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. To illustrate this, we have the following example.

Example 1.6. Consider a Bernoulli random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with probability $P\{X_n = 1\} = 1/n$, and another random sequence $Y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined as $Y_n \triangleq nX_n$ for $n \in \mathbb{N}$. Then, $P\{Y_n = 0\} = 1 - 1/n$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. So $\mathbb{E}[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $\frac{m_t}{t}$ converging to $\frac{1}{\mu}$. We first need this technical Lemma.

Proposition 1.7 (Wald's Lemma for renewal process). *Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be the renewal function for a renewal counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with* i.i.d. *inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having finite mean $\mu = \mathbb{E}[X_1] < \infty$. Then, $N_t + 1$ is a stopping time for the sequence X, and*

$$\mathbb{E}\left[\sum_{i=1}^{N_t+1} X_i\right] = \mu(1 + m_t).$$

Proof. We observe that for any $n \in \mathbb{N}$, the event $\{N_t + 1 = n\}$ belongs to $\sigma(X_1, \dots, X_n)$, since

$$\{N_t + 1 = n\} = \{S_{n-1} \leqslant t < S_n\} = \left\{\sum_{i=1}^{n-1} X_i \leqslant t < \sum_{i=1}^{n-1} X_i + X_n\right\} \in \sigma(X_1, \dots, X_n).$$

Thus $N_t + 1$ is a stopping time with respect to the random sequence X, and the result follows from Wald's Lemma. ☐

Theorem 1.8 (Elementary renewal theorem). *For a renewal process with finite mean inter-arrival times, the renewal function satisfies*

$$\lim_{t\to\infty} \frac{m_t}{t} = \frac{1}{\mu}. \tag{4}$$

Proof. By the assumption, we have mean $\mu < \infty$. Further, we know that $S_{N_t+1} > t$. Taking expectations on both sides and using Proposition 1.7, we have $\mu(m_t + 1) > t$. Dividing both sides by μt and taking lim inf on both sides, we get

$$\liminf_{t\to\infty} \frac{m_t}{t} \geqslant \frac{1}{\mu}. \tag{5}$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $(\bar{X}_n = \min(X_n, M) : n \in \mathbb{N})$ with mean denoted by μ_M. These modified inter-arrival times are *i.i.d.* and hence we can define the corresponding renewal process $(\bar{S}_n = \sum_{i=1}^n \bar{X}_i : n \in \mathbb{N})$ and the associated counting process $\bar{N}_t = \sum_{n\in\mathbb{N}} 1_{\{\bar{S}_n \leqslant t\}}$. Note that since $S_n \geqslant \bar{S}_n$, the number of arrivals would be higher for renewal process $\bar{N}_t$ with truncated random variables. That is,

$$N_t \leqslant \bar{N}_t. \tag{6}$$

Further, due to truncation of inter-arrival time, next renewal happens within M units of time, that is $\bar{S}_{\bar{N}_t+1} \leqslant t + M$. Taking expectations on both sides in the above equation, using Proposition 1.7, dividing both sides by $t\mu_M$, and taking lim sup on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m}_t}{t} \leqslant \frac{1}{\mu_M}.$$

Recognizing that $\lim_{M\to\infty} \mu_M = \mu$, the result follows from taking expectations on both sides of (6), and the lower bound on lim inf on the ratio m_t/t. □

Corollary 1.9. *For a delayed renewal process with finite inter-arrival durations, we have* $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.10 (Markov chain). Consider a positive recurrent discrete time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ taking values in a discrete set $\mathcal{X} \subset \mathbb{R}$. Let the initial state be $X_0 = x \in \mathcal{X}$ and $\tau_y^+(0) = 0$ for $y \neq x \in \mathcal{X}$, then we can inductively define the nth recurrent time to state y as a stopping time

$$\tau_y^+(n) = \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$

Since any discrete time Markov chain satisfies the strong Markov property, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ form a delayed renewal process with the first arrival distribution $P_x\left\{\tau_y^+(1) = k\right\} = f_{xy}^{(k)}$, and the common distribution of the inter-arrival duration $X_n, n \geqslant 2$ in terms of first return probability as

$$P_y\left\{\tau_y^+(1) = k\right\} = f_{yy}^{(k)},\ k \in \mathbb{N}.$$

We denote the associated counting process by $N_y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, where $N_y(n) = \sum_{i\in\mathbb{N}} \mathbb{1}_{\{\tau_y^+(i)\leqslant n\}} = \sum_{k=1}^n \mathbb{1}_{\{X_k = y\}}$ denotes the number of visits to state y up to time n. Let $\mu_{yy} = \mathbb{E}_y \tau_y^+(1)$ be the finite mean inter-arrival time for the renewal process, also the mean recurrence time to state y. From the strong law for delayed renewal processes it follows that

$$P_y\left\{\lim_{n\in\mathbb{N}} \frac{N_y(n)}{n} = \frac{1}{\mu_{yy}}\right\} = 1.$$

Since $N_y(n)$ is number of visits to state y in first n time steps, we have $\mathbb{E}_x N_y(n) = \sum_{k=1}^n P_x\{X_k = y\} = \sum_{k=1}^n p_{xy}^{(k)}$ From the basic renewal theorem for delayed renewal process it follows that

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \lim_{n\in\mathbb{N}} \frac{\mathbb{E}_x[N_y(n)]}{n} = \frac{1}{\mu_{yy}}.$$

1.3 Central limit theorem for renewal processes

Theorem 1.11. *For a renewal process with inter-arrival times having finite mean μ and finite variance σ^2, the associated counting process converges to a normal random variable in distribution. Specifically,*

$$\lim_{t\to\infty} P\left\{\frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx.$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N_t < u\} = \{S_u > t\}$. By equating probability measures on both sides, we get

$$P\{N_t < u\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t - u\mu}{\sigma\sqrt{u}}\right\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-1/2}\right\}.$$

By central limit theorem, $\frac{S_u - u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. We also observe that

$$\lim_{t\to\infty} -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^{-1/2} = -y.$$

These results combine with the symmetry of normal random variable to give us the result. □