Lecture-12: Key Renewal Theorem

1 Key Renewal Theorem

Theorem 1.1 (Key renewal theorem). *Consider a recurrent renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with renewal function $m : \mathbb{R}_+ \to \mathbb{R}_+$, the common mean μ, and the distributionF for* i.i.d. *inter-renewal times. For any directly Riemann integrable function $z \in \mathbb{D}$, we have*

$$\lim_{t\to\infty} \int_0^t z(t-x)dm(x) = \begin{cases} \frac{1}{\mu}\int_0^\infty z(t)dt, & F \text{ is non-lattice},\\ \frac{d}{\mu}\sum_{k\in\mathbb{Z}_+} z(t+kd), & F \text{ is lattice with period } d, \quad t = nd. \end{cases}$$

Proposition 1.2 (Equivalence). *Blackwell's theorem and key renewal theorem are equivalent.*

Proof. Let's assume key renewal theorem is true. We select $z : \mathbb{R}_+ \to \mathbb{R}_+$ as a simple function with value unity on interval $[0,a]$ for $a \geqslant 0$ and zero elsewhere. That is, $z(t) = \mathbb{1}_{[0,a]}(t)$ for any $t \in \mathbb{R}_+$. From Proposition A.3, it follows that z is directly Riemann integrable. Therefore, by Key Renewal Theorem, we have

$$\lim_{t\to\infty}[m(t) - m(t-a)] = \frac{a}{\mu}.$$

We defer the formal proof of converse for a later stage. We observe that, from Blackwell theorem, it follows

$$\lim_{t\to\infty}\frac{dm(t)}{dt} \overset{(a)}{=} \lim_{a\to 0}\lim_{t\to\infty}\frac{m(t+a)-m(t)}{a} = \frac{1}{\mu}.$$

where in (a) we can exchange the order of limits under certain regularity conditions. □

Remark 1. Key renewal theorem is very useful in computing the limiting value of some function $g(t)$, probability or expectation of an event at an arbitrary time t, for a renewal process. This value is computed by conditioning on the time of last renewal prior to time t.

Corollary 1.3 (Delayed key renewal theorem). *Consider an aperiodic and recurrent delayed renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with independent inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with first inter-renewal time distribution G and common inter-renewal time distribution F for $(X_n : n \geqslant 2)$. Let the renewal function be denoted by $m_D(t)$ and means $\mathbb{E}X_1 = \mu_G$ and $\mathbb{E}X_2 = \mu_F$. For any directly Riemann integrable function $z \in \mathbb{D}$ and F non-lattice, we have*

$$\lim_{t\to\infty}\int_0^t z(t-x)dm_D(x) = \frac{1}{\mu_F}\int_0^\infty z(t)dt.$$

Remark 2. Any kernel function $K(t) = P\{Z_t \in A, X_1 > t\} \leqslant \bar{F}(t)$, and hence is d.R.i. from Proposition A.3(b).

Example 1.4 (Limiting distribution of regenerative process). For a regenerative process Z over a delayed renewal process S with finite mean *i.i.d.* inter-arrival times, we have $K_2(t) = P\{Z_{S_1+t} \in A, X_2 > t\} \leqslant \bar{F}(t)$ for any $A \in \mathcal{B}(\mathbb{R})$, and hence the kernel function $K_2 \in \mathbb{D}$. Applying Key Renewal Theorem to renewal function, we get the limiting probability of the event $\{Z_t \in A\}$ as

$$\lim_{t\to\infty} P\{Z_t \in A\} = \lim_{t\to\infty}(m_D * K_2)(t) = \frac{1}{\mu_F}\int_{t=0}^x K_2(t)dt.$$

Example 1.5 (Limiting distribution of age and excess time). For a delayed renewal process S with finite mean independent inter-renewal times such that the distribution of first renewal time is G, and the distribution of subsequent renewal times are identically F. Denoting the associated counting process by N_D and renewal function m_D, we can write the limiting probability distribution of age as $F_e(x) \triangleq \lim_{t\to\infty} P\{A(t) \leqslant x\}$. We can write the complementary distribution as

$$\bar{F}_e(x) = \lim_{t\to\infty} P\{A(t) \geqslant x\} = \lim_{t\to\infty} \int_0^t dm_D(t-y)\mathbb{1}_{\{y\geqslant x\}}\bar{F}(y) = \frac{1}{\mu_F}\int_x^\infty \bar{F}(y)dy.$$

Example 1.6 (Limiting on probability of alternating renewal process). Consider an alternating renewal process W with random on and off time sequence Z and Y respectively, such that (Z, Y) is *i.i.d.* . We denote the distribution of on and off times by non-lattice functions H and G respectively. If $\mathbb{E}Z_n$ and $\mathbb{E}Y_n$ are finite, then applying Key renewal theorem to the limiting probability of alternating process being on, we get

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}(m * \bar{H})(t) = \frac{\mathbb{E}Z_n}{\mathbb{E}Z_n + \mathbb{E}Y_n}.$$

A Directly Riemann Integrable

For each scalar $h > 0$ and natural number $n \in \mathbb{N}$, we can define intervals $I_n(h) \triangleq [(n-1)h, nh)$, such that the collection $(I_n(h), n \in \mathbb{N})$ partitions the positive real-line $\mathbb{R}_+$. For any function $z : \mathbb{R}_+ \to \mathbb{R}_+$ be a function bounded over finite intervals, we can denote the infimum and supremum of z in the interval I_n as

$$\underline{z}_h(n) \triangleq \inf\{z(t) : t \in I_n(h)\} \qquad \bar{z}_h(n) \triangleq \sup\{z(t) : t \in I_n(h)\}.$$

We can define functions $\underline{z}_h, \bar{z}_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\underline{z}_h(t) \triangleq \sum_{n\in\mathbb{N}} \underline{z}_h(n)\mathbb{1}_{I_n(h)}(t)$ and $\bar{z}_h(t) \triangleq \sum_{n\in\mathbb{N}} \bar{z}_h(n)\mathbb{1}_{I_n(h)}(t)$ for all $t \in \mathbb{R}_+$. From the definition, we have $\underline{z}_h \leqslant z \leqslant \bar{z}_h$ for all $h \geqslant 0$. The infinite sums of infimum and supremums over all the intervals $(I_n(h), n \in \mathbb{N})$ are denoted by

$$\int_{t\in\mathbb{R}_+} \bar{z}_h(t)dt = h\sum_{n\in\mathbb{N}} \bar{z}_h(n), \qquad \int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt = h\sum_{n\in\mathbb{N}} \underline{z}_h(n).$$

Remark 3. Since $\underline{z}_h \leqslant z \leqslant \bar{z}_h$, we observe that $\int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt \leqslant \int_{t\in\mathbb{R}_+} z(t)dt \leqslant \int_{t\in\mathbb{R}_+} \bar{z}_h(t)dt$. If both left and right limits exist and are equal, then the integral value $\int_{t\in\mathbb{R}_+} z(t)dt$ is equal to the limit.

Definition A.1 (directly Riemann integrable (d.R.i.)). A function $z : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is **directly Riemann integrable** and denoted by $z \in \mathbb{D}$ if the partial sums obtained by summing the infimum and supremum of h, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

$$\sum_{n\in\mathbb{N}} h\bar{z}_h(n) < \infty, \qquad \lim_{h\downarrow 0}\int_{t\in\mathbb{R}_+} \bar{z}_h(t)dt = \lim_{h\downarrow 0}\int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt.$$

The limit is denoted by $\int_{t\in\mathbb{R}_+} z(t)dt = \lim_{h\downarrow 0}\sum_{n\in\mathbb{N}} h\bar{z}_h(n) = \lim_{h\downarrow 0}\sum_{n\in\mathbb{N}} h\underline{z}_h(n)$. For a real function $z : \mathbb{R}_+ \to \mathbb{R}$, we can define the positive and negative parts by $z^+, z^- : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ $z^+(t) \triangleq z(t) \vee 0$, and $z^-(t) \triangleq -(z(t) \wedge 0)$. If both $z^+, z^- \in \mathbb{D}$, then $z \in \mathbb{D}$ and the limit is

$$\int_{\mathbb{R}_+} z(t)dt \triangleq \int_{\mathbb{R}_+} z^+(t)dt - \int_{\mathbb{R}_+} z^-(t)dt.$$

Remark 4. We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive M, a function $z : [0, M] \to \mathbb{R}$ is Riemann integrable if

$$\lim_{h\to 0}\int_0^M \underline{z}_h(t)dt = \lim_{h\to 0} h\int_0^M \underline{z}_h(t)dt.$$

In this case, the limit is the value of the integral $\int_0^M z(t)dt$. For a function $z : \mathbb{R}_+ \to \mathbb{R}$,

$$\int_{t\in\mathbb{R}_+} z(t)dt = \lim_{M\to\infty} \int_0^M z(t)dt,$$

if the limit exists. For many functions, this limit may not exist.

Remark 5. A directly Riemann integrable function over $\mathbb{R}_+$ is also Riemann integrable, but the converse need not be true. For instance, for $E_n \triangleq \left[n - \frac{1}{2n^2}, n + \frac{1}{2n^2}\right]$ for each $n \in \mathbb{N}$, consider the following Riemann integrable function $z : \mathbb{R}_+ \to \mathbb{R}_+$

$$z(t) = \sum_{n\in\mathbb{N}} \mathbb{1}_{E_n}(t), \qquad t \in \mathbb{R}_+.$$

We observe that z is Riemann integrable, however $\int_{t\in\mathbb{R}_+} \overline{z}(t)dt$ is always infinite. It suffices to show that $h\sum_{n\in\mathbb{N}} \overline{z}_h(n)$ is always infinite for every $h > 0$. Since the collection $\{I_n(h) : n \in N\}$ partitions the entire $\mathbb{R}_+$, for each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $E_n \cap I_m(h) \neq \emptyset$, and therefore $\overline{z}_m(h) = 1$. It follows that

$$\int_{t\in\mathbb{R}_+} \overline{z}(t)dt = \sum_{m\in\mathbb{N}} h = \infty.$$

Proposition A.2 (Necessary conditions for d.R.i.). *If a function $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable, then z is bounded and continuous a.e.*

Proposition A.3 (Sufficient conditions for d.R.i.). *A function $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable, if any of the following conditions hold.*

(a) z is monotone non-increasing, and Lebesgue integrable.

(b) z is bounded above by a directly Riemann integrable function.

(c) z has bounded support.

(d) $\int_{t\in\mathbb{R}_+} \overline{z}_h dt$ is bounded for some $h > 0$.

Proposition A.4 (Tail Property). *If $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable and has bounded integral value, then $\lim_{t\to\infty} z(t) = 0$.*

Corollary A.5. *Any distribution $F : \mathbb{R}_+ \to [0, 1]$ with finite mean μ, the complementary distribution function $\bar{F}$ is d.R.i.*

Proof. Since $\bar{F}$ is monotonically non-increasing and its Lebesgue integration is $\int_{\mathbb{R}_+} \bar{F}(t)dt = \mu$, the result follows from Proposition A.3(a). □

INCLUDE THIS AFTER HOMEWORK 2

A.1 Age and excess times

Consider a renewal process with renewal instants $\{S_n : n \in \mathbb{N}\}$, and *i.i.d.* inter-renewal times $\{X_n : n \in \mathbb{N}\}$ with the common non-lattice distribution F. At time t, the last renewal occurred at time $S_{N(t)}$, and the next renewal will occur at time $S_{N(t)+1}$. Recall that the age $A(t)$ is the time since the last renewal and the excess time $Y(t)$ is the time till the next renewal.

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That is,

$$A(t) = t - S_{N(t)}, \qquad Y(t) = S_{N(t)+1} - t.$$

We are interested in finding the limiting distribution of the age and excess time. That, is for a fixed x, we wish to compute

$$\lim_{t\to\infty} \Pr\{A(t) \leq x\}, \qquad \lim_{t\to\infty} \Pr\{Y(t) \leq x\}.$$

Proposition A.6. *Limiting age distribution for a renewal process with non-lattice distribution F is*

$$\lim_{t\to\infty} \Pr\{A(t) \leq x\} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt. \tag{1}$$

Proof. We will call this renewal process to be on, when the age is less than x. That is, we consider an alternative renewal process W such that

$$W(t) = 1\{A(t) \leq x\}. \tag{2}$$

This is an alternating renewal process with finite probability of off times being zero. Further, we can write the nth on time Z_n for this renewal process as

$$Z_n = \min\{x, X_n\}. \tag{3}$$

From limiting on probability of alternating renewal process, we get

$$\lim_{t\to\infty} \Pr\{A(t) \leq x\} = \lim_{t\to\infty} P\{W(t) = 1\} = \frac{\mathbb{E}\min\{x, X\}}{\mathbb{E}X} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt. \tag{4}$$

□

Alternative proof. Another way of evaluating $\lim_{t\to\infty} \Pr\{A(t) \leq x\}$ is to note that $\{A(t) \leq x\} = \{S_{N(t)} \geq t - x\}$. From the distribution of $S_{N(t)}$ and the fact that the support of renewal function $m(t)$ is positive real life,

$$\Pr\{A(t) \leq x\} = \Pr\{S_{N(t)} \geq t - x\} = \int_{t-x}^{\infty} \bar{F}(t-y)dm(y) = \int_{-\infty}^{x} \bar{F}(y)dm(t-y) = \int_0^x \bar{F}(u)dm(t-u). \tag{5}$$

Applying key renewal theorem, we get the result. □

We see that the limiting distribution of age and excess times are identical. This can be observed by noting that if we consider the reversed processes (an identically distributed renewal process), $Y(t)$, the "excess life time" at t is same as the age at t, $A(t)$ of the original process.

Proposition A.7. *Limiting excess time distribution for a renewal process with non-lattice distribution F is*

$$\lim_{t\to\infty} \Pr\{Y(t) \leq x\} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt. \tag{6}$$

Proof. We can repeat the same proof for limiting age distribution, where we obtain an alternative renewal process by defining on times when the excess time is less than x. That is, we consider an alternative renewal process W such that

$$W(t) = 1\{Y(t) \leq x\}. \tag{7}$$

□

Corollary A.8. *Limiting mean excess time for a renewal process with* i.i.d. *inter-renewal times $\{X_n : n \in \mathbb{N}\}$ having non-lattice distribution F and mean μ is*

$$\lim_{t\to\infty} \mathbb{E}Y(t) = \frac{\mathbb{E}[X^2]}{2\mu}. \tag{8}$$

Proof. One can get the limiting mean from the limiting distribution by integrating its complement. This involves exchanging limit and integration, which can be justified using monotone convergence theorem. Hence, exchanging integrals using Fubini's theorem and integrating by parts, we get

$$\lim_{t\to\infty} \mathbb{E}Y(t) = \int_0^\infty \lim_{t\to\infty} P\{Y(t) > x\}dx = \frac{1}{\mu}\int_0^\infty \int_x^\infty \bar{F}(t)dt = \frac{1}{2\mu}\int_0^\infty \bar{F}(t)dt^2 = \frac{1}{2\mu}\int_0^\infty t^2 dF(t). \tag{9}$$

Alternatively, one can derive it directly from the regenerative process theory. □

Lemma A.9. *Limiting empirical time average of excess time for a renewal process with* i.i.d. *inter-renewal times* $\{X_n : n \in \mathbb{N}\}$ *having non-lattice distribution F and mean μ is almost surely*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)du = \frac{\mathbb{E}X^2}{2\mu}. \tag{10}$$

Proof. Recall that excess times are linearly decreasing in each renewal duration, with value X_n to 0 in nth renewal duration of length X_n. Conditioned on $N(t)$, one can write

$$\int_0^t Y(u)du = \frac{1}{2}\sum_{n=1}^{N(t)} X_n^2 + \int_{S_{N(t)}}^t (S_{N(t)+1} - u)du. \tag{11}$$

In particular, one can write the following

$$\frac{\sum_{n=1}^{N(t)} X_n^2}{2N(t)}\left(\frac{N(t)}{t}\right) \le \frac{1}{t}\int_0^t Y(u)du \le \frac{\sum_{n=1}^{N(t)+1} X_n^2}{2(N(t)+1)}\left(\frac{N(t)+1}{t}\right). \tag{12}$$

Result follows from strong law of large number by taking limits on both sides. □

A.2 The Inspection Paradox

Define $X_{N(t)+1} = A(t) + Y(t)$ as the length of the renewal interval containing t, in other words, the length of current renewal interval.

Theorem A.10 (inspection paradox). *For any x, the length of the current renewal interval to be greater than x is more likely than that for an ordinary renewal interval with distribution function F. That is,*

$$P\{X_{N(t)+1} > x\} \ge \bar{F}(x). \tag{13}$$

Proof. Conditioning on the joint distribution of last renewal instant and number of renewals, we can write

$$\Pr\{X_{N(t)+1} > x\} = \int_0^t \Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} dF_{(S_{N(t)}, N(t))}. \tag{14}$$

Now we have,

$$\Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} = \Pr\{X_{n+1} > x | X_{n+1} > t - y\} = \frac{\Pr\{X_{n+1} > \max(x, t-y)\}}{\Pr\{X_{n+1} > t - y\}}. \tag{15}$$

Applying Chebyshev's sum inequality to increasing positive functions $f(z) = 1\{t > x\}$ and $g(z) = 1\{z > t - y\}$, we get

$$\mathbb{E}f(X_{n+1})g(X_{n+1})/\mathbb{E}g(X_{n+1}) \ge \mathbb{E}f(X_{n+1}) = \bar{F}(x). \tag{16}$$

We get the result by integrating over the joint distribution. □

We also have a weaker version of inspection paradox involving the limiting distribution of $X_{N(t)+1}$.

Lemma A.11. *For any x, the limiting probability of length of the current renewal interval being greater than x is larger than the corresponding probability of an ordinary renewal interval with distribution function F. That is,*

$$\lim_{t\to\infty} P\{X_{N(t)+1} > x\} \ge \bar{F}(x).$$

Proof. Consider an alternating renewal process W for which the on time is the renewal duration if greater than x, and zero otherwise. That is,

$$W(t) = 1\{X_{N(t)+1} > x\}. \tag{17}$$

Hence, each renewal duration consists of either on or off intervals, depending if the renewal duration length is greater than x or not. We can denote nth on and off times by Z_n and Y_n respectively, where

$$Z_n = X_n 1\{X_n > x\}, \qquad\qquad Y_n = X_n 1\{X_n \le x\}.$$

From the definition, we have

$$\mathbb{E}W(t) = P\{X_{N(t)+1} > x\} = P\{\text{ on at time } t\}. \tag{18}$$

From the alternating renewal process theorem, we conclude that

$$\lim_{t\to\infty} \Pr\{X_{N(t)+1} > x\} = \frac{\mathbb{E}X1\{X > x\}}{\mathbb{E}X}. \tag{19}$$

The result follows from Chebyshev's sum inequality applied to positive increasing function $f(z) = z$ and $g(z) = 1\{z > x\}$. □

The inspection paradox states, in essence, that if we pick a point t, it is more likely that an inter-renewal interval with larger length will contain t than the smaller ones. For instance, if X_i were equally likely to be ϵ or $1-\epsilon$, we see that the mean of any inter arrival length is 1 for any value of $\epsilon \in (0,1)$. However, for small values ϵ, it is more likely that a given t will be in an interval of length $1-\epsilon$ than in an interval of length ϵ.

Proposition A.12. *If the inter arrival time is non-lattice and* $\mathbb{E}[X^2] < \infty$, *we have*

$$\lim_{t\to\infty} \left(m(t) - \frac{t}{\mu} \right) = \frac{\mathbb{E}[X^2]}{2\mu^2} - 1. \tag{20}$$

Proof. From definition of excess time $Y(t)$ and Wald's lemma for stopping time $N(t)+1$ for renewal processes, it follows

$$\mathbb{E}S_{N(t)+1} = \mathbb{E}[\sum_{i=1}^{N(t)+1} X_n] = \mu(m(t)+1) = t + \mathbb{E}[Y(t)].$$

□