Lecture-17: Continuous Time Markov Chains

1 Markov Process

Definition 1.1. For any stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ indexed by positive reals and taking values in $\mathcal{X} \subseteq \mathbb{R}$, the history of the process until time $t \in \mathbb{R}_+$ is the collection of all the events that can be determined by the realization of the process X until time t, denoted by $\mathcal{F}_t \triangleq \sigma(X_u,\ u \leqslant t)$.

Definition 1.2. A real-valued stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ indexed by positive reals, and with state space $\mathcal{X}$, is a **Markov process** if it satisfies the Markov property. That is for any Borel measurable set $A \in \mathcal{B}(\mathcal{X})$, the distribution of the future states conditioned on the present, is independent of the past, and

$$P(\{X_{t+s} \in A\} | \mathcal{F}_s) = P(\{X_{t+s} \in A\} | \sigma(X_s)), \text{ for all } s,t \in \mathbb{R}_+.$$

Definition 1.3. A Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with countable state space $\mathcal{X}$ is called **continuous time Markov chain (CTMC)**.

Remark 1. The Markov property for the CTMCs can be interpreted as follows. For all times $0 < t_1 < \cdots < t_m < t$ and states $x_1, \dots, x_m, y \in \mathcal{X}$, we have

$$P(\{X_t = y\} \mid \cap_{k=1}^m \{X_{t_k} = x_k\}) = P(\{X_t = y\} \mid \{X_{t_m} = x_m\}).$$

Example 1.4 (Counting process). Any simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with independent increments is a CTMC. This implies any (possibly time-inhomogeneous) Poisson process is a CTMC. Countability of the state space is clear from the definition of the counting process. For Markov property, we observe that for $t > s$, the increment $N_t - N_s$ is independent of $\mathcal{F}_s$. Hence for the natural filtration $\mathcal{F}_\bullet$,

$$\mathbb{E}[\mathbb{1}_{\{N_t=j\}} \mid \mathcal{F}_s] = \sum_{i \in \mathbb{Z}_+} \mathbb{E}[\mathbb{1}_{\{N_t=j, N_s=i\}} \mid \mathcal{F}_s] = \sum_{i \in \mathbb{Z}_+} \mathbb{1}_{\{N_s=i\}} \mathbb{E}\mathbb{1}_{\{N_t - N_s = j-i\}} = \mathbb{E}[\mathbb{1}_{\{N_t=j\}} \mid \sigma(N_s)].$$

1.1 Transition probability kernel

Definition 1.5. We define the **transition probability** from state x at time s to state y at time $t+s$ as

$$P_{xy}(s, s+t) \triangleq P(\{X_{s+t} = y\} \mid \{X_s = x\}).$$

Definition 1.6. The Markov process has **homogeneous** transitions for all states $x, y \in \mathcal{X}$ and all times $s, t \in \mathbb{R}_+$, if

$$P_{xy}(t) \triangleq P_{xy}(0,t) = P_{xy}(s, s+t).$$

We denote the **transition probability kernel/function** at time t by $P(t) \triangleq (P_{xy}(t) : x, y \in \mathcal{X})$.

Remark 2. We will mainly be interested in continuous time Markov chains with homogeneous jump transition probabilities. We will assume that the sample path of the process X is right continuous with left limits at each time $t \in \mathbb{R}_+$.

Remark 3. Conditioned on the initial state of the process is x, we denote the conditional probability for any event $A \in \mathcal{F}$ as $P_x(A) \triangleq P(A \mid \{X_0 = x\})$ and the conditional expectation for any random variable $Y : \Omega \to \mathbb{R}$ as $\mathbb{E}_x Y \triangleq \mathbb{E}[Y \mid \{X_0 = x\}]$.

Lemma 1.7 (stochasticity). *Transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X} \times \mathcal{X}}$ at each time $t \in \mathbb{R}_+$ is a stochastic matrix.*

Proof. From the countable partition of the state space $\mathcal{X}$, we can write $1 = P_x(\{X_t \in \mathcal{X}\}) = \sum_{y\in\mathcal{X}} P_{xy}(t)$ for any state $x \in \mathcal{X}$. □

Lemma 1.8 (semigroup). *Transition kernel satisfies the semigroup property, i.e. $P(s+t) = P(s)P(t)$ for all $s,t \in \mathbb{R}_+$.*

Proof. From the Markov property and homogeneity of CTMC, and law of total probability, we can write the (x,y)th entry of kernel matrix $P(s+t)$ as

$$P_{xy}(s+t) = P_{xy}(0,s+t) = \sum_{z\in\mathcal{X}} P_{xz}(0,s)P_{zy}(s,s+t) = \sum_{z\in\mathcal{X}} P_{xz}(0,s)P_{zy}(0,t) = [P(s)P(t)]_{xy}.$$

Result follows since states $x,y \in \mathcal{X}$ were chosen arbitrarily. □

Lemma 1.9 (continuity). *Transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ for a homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is a continuous function of time $t \in \mathbb{R}_+$, such that $\lim_{t\downarrow 0} P(t) = I$, the identity matrix. That is, $P_{xx}(0) = 1$ and $P_{xy}(0) = 0$ for all $y \neq x \in \mathcal{X}$.*

Proof. We will first show the continuity of transition kernel at time $t = 0$. From right continuity of sample paths for process X, we have $\lim_{t\downarrow 0} X_t = X_0$ and from continuity of probability functions we get $\lim_{t\downarrow 0} P_x \{X_t = y\} = P_x(\lim_{t\downarrow 0} X_t = y) = I_{xy}$.

For continuity at any time $t > 0$, we can write the difference $P(t+h) - P(t) = P(t)(P(h) - I)$ using the semigroup property of the transition kernel. The continuity of transition kernel at time $t = 0$, and boundedness of $P(t)$ implies continuity of $P(t)$ at all times $t > 0$. □

Remark 4. Since each entry of transition kernel $P(t)$ is a probability, semigroup property leads to characterization of the kernel $P(t)$ completely.

Proposition 1.10. *For a time-homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with transition kernel P, for all times $0 < t_1 < \cdots < t_m$ and states $x_0, x_1, \ldots, x_m \in \mathcal{X}$, we have*

$$P(\cap_{k=1}^m \{X_{t_k} = x_k\} \mid \{X_0 = x_0\}) = P_{x_0x_1}(t_1)P_{x_1x_2}(t_2 - t_1)\ldots P_{x_{m-1}x_m}(t_m - t_{m-1}).$$

Corollary 1.11. *All finite dimensional distributions of the CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is governed by the initial distribution.*

Proof. Let $\nu_0 \in \mathcal{M}(\mathcal{X})$ be the initial distribution of the CTMC X, such that $\nu_0(x_0) = P\{X_0 = x_0\}$ for each $x_0 \in \mathcal{X}$. For all finite index sets $F \subset \mathbb{R}_+, |F| = m$ and state vector $x \in \mathcal{X}^m$, we have

$$P(\cap_{t_j\in F} \left\{X_{t_j} = x_j\right\}) = \sum_{x_0\in\mathcal{X}} \nu_0(x_0)P_{x_0x_1}(t_1)\ldots P_{x_{m-1}x_m}(t_m - t_{m-1}).$$

□

Definition 1.12 (Exponentiation of a matrix). For a matrix A with spectral radius less than unity, we can define

$$e^A \triangleq I + \sum_{n\in\mathbb{N}} \frac{A^n}{n!}.$$

Lemma 1.13. *For a homogeneous CTMC, we can write the transition kernel $P(t) = e^{tQ}$ in terms of a constant matrix $e^Q \triangleq P(1)$.*

Proof. This follows from the semigroup property and the continuity of transition kernel $P(t)$. In particular, we notice that $P(n) = P(1)^n$ and $P(\frac{1}{m}) = P(1)^{\frac{1}{m}}$ for all $m,n \in \mathbb{N}$. Since, any rational number $q \in \mathbb{Q}$ can be expressed as a ratio of integers with no common divisor, we get

$$P(q) = P(1)^q, \quad q \in \mathbb{Q}.$$

Since the rationals are dense in reals and P is continuous function, it follows that $P(t) = P(1)^t$ for all $t \in \mathbb{R}$ and the result follows from definition of $e^Q = P(1)$. □

1.2 Excess time in a state

Definition 1.14. From the definition of excess time as the time until next transition, we can write the excess time at time $t \in \mathbb{R}_+$ for the CTMC X as

$$Y_t \triangleq \inf\{s > 0 : X_{t+s} \neq X_t\}.$$

Remark 5. We observe that Y_t is the excess remaining time the process spends in state X_t at instant t. That is, $X_{t+Y_t} \neq X_t$.

Remark 6. For a homogeneous CTMC X, the distribution of excess time Y_t conditioned on the current state X_t, doesn't depend on time t. Hence, we can define the following conditional complementary distribution of excess time as $\bar{F}_x(u) \triangleq P(\{Y_t > u\} \mid \{X_t = x\}) = P_x\{Y_0 > u\}$.

Lemma 1.15. *For a homogeneous CTMC X, there exists a positive sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, such that*

$$\bar{F}_x(u) \triangleq P(\{Y_t > u\} \mid \{X_t = x\}) = e^{-u\nu_x}, \quad x \in \mathcal{X}.$$

Proof. We fix a state $x \in \mathcal{X}$, and observe that the function $\bar{F}_x \in [0,1]$ is non-negative, non-increasing, and right-continuous in u. Using the Markov property and the time-homogeneity, we can show that $\bar{F}_x$ satisfies the semigroup property. In particular,

$$\bar{F}_x(u+v) = P(\{Y_t > u+v\} \mid \{X_t = x\}) = P(\{Y_t > u, Y_{t+u} > v\} \mid \{X_t = x\}) = \bar{F}_x(u)\bar{F}_x(v).$$

The only continuous function $\bar{F}_x \in [0,1]$ that satisfies semigroup property is an exponential function with a negative exponent. □

Definition 1.16. For a CTMC X, a state $x \in \mathcal{X}$ is called

(i) **absorbing** if $\nu_x = 0$,

(ii) **stable** if $\nu_x \in (0, \infty)$, and

(iii) **instantaneous** if $\nu_x = \infty$.

Remark 7. The sojourn time in an absorbing state is ∞, zero in an instantaneous state, and almost surely finite and non-zero in a stable state.

Definition 1.17. A homogeneous CTMC with no instantaneous states is called a **pure jump** CTMC. A pure jump CTMC with

(i) all stable states and $\inf_{x\in\mathcal{X}} \nu_x \geqslant \nu > 0$ is called **stable**, and

(ii) $\sup_{x\in\mathcal{X}} \nu_x \leqslant \nu < \infty$ is called **regular**.

Remark 8. Pure jump homogeneous CTMC with finite stable states are stable and regular. We will focus on pure jump homogeneous CTMC over countably infinite states, that are stable and regular.

Example 1.18 (Poisson process). Consider the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for a Poisson point process with homogeneous rate λ. Using the stationary independent increment property, we have for all $u \geqslant 0$

$$\bar{F}_i(u) = P(\{Y_t > u\} \mid \{N_t = i\}) = P(\{N_{t+u} = i\} \mid \{N_t = i\}) = P\{N_{t+u} - N_t = 0\} = P\{Y_t > u\} = e^{-\lambda u}.$$

A Poisson process with finite non-zero rate is a pure-jump CTMC with stable states.

1.3 Strong Markov property

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a continuous filtration $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in \mathbb{R}_+)$ defined on this space.

Definition 1.19. A random variable $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a **stopping time** with respect to $\mathcal{F}_\bullet$ if $\{\tau \leqslant t\} \in \mathcal{F}_t$ for each $t \in \mathbb{R}_+$. That is, a random variable τ is a stopping time if the event $\{\tau \leqslant t\}$ can be determined completely by the history $\mathcal{F}_t$. An almost surely finite stopping time τ is called **proper**.

Definition 1.20. A stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ adapted to filtration $\mathcal{F}_\bullet$ has **strong Markov property** if for any proper stopping time τ with respect to $\mathcal{F}_\bullet$, and set $A \in \mathcal{B}(\mathcal{X})$, we have

$$P(\{X_{\tau+s} \in A\} \mid \mathcal{F}_\tau) = P(\{X_{\tau+s} \in A\} \mid \sigma(X_\tau)).$$

Lemma 1.21. *A continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ has the strong Markov property.*

Proof. It follows from the right continuity of the CTMC process X, and the fact that the map $t \mapsto \mathbb{E}[f(X_{s+t}) \mid \sigma(X_t)]$ is right-continuous for any bounded continuous function $f : \mathcal{X} \to \mathbb{R}$. To see the right continuity of the map, we observe that

$$\mathbb{E}[f(X_{s+t}) \mid \sigma(X_t)] = \sum_{x \in \mathcal{X}} \mathbb{1}_{\{X_t = x\}} \sum_{y \in \mathcal{X}} P_{xy}(s) f(y).$$

Right-continuity of the map follows from the right continuity of the sample paths of process X, right-continuity and boundedness of the kernel function, and boundedness and continuity of f, and bounded convergence theorem. □

Corollary 1.22. *A pure jump CTMC X satisfies the following strong Markov property. For any proper stopping time τ, finite $m \in \mathbb{N}$, finite times $0 < t_1 < \cdots < t_m$, any event $H \in \mathcal{F}_\tau$ and states $x_0, x_1, \dots, x_m \in \mathcal{X}$, we have*

$$P(\cap_{k=1}^m \{X_{t_k+\tau} = x_k\} \mid H \cap \{X_\tau = x_0\}) = P_{x_0}(\cap_{k=1}^m \{X_{t_k} = x_k\}).$$

Remark 9. In particular for a pure-jump time-homogeneous CTMC X, proper stopping time τ, and event $H \in \mathcal{F}_\tau$, we have

$$P(\{X_{\tau+s} = y\} \mid \{X_\tau = x\} \cap H) = P_{xy}(s).$$