Lecture-19: Markov Processes: Stationarity

1 Generator Matrix

From Lemma **??** for a homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, we can write the probability transition kernel function $P(t) = e^{tQ}$, where $e^Q = P(1)$. The matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ is called the **generator matrix** for the homogeneous CTMC X. From the Definition **??** for the exponentiation of matrix, this implies that

$$P(t) = I + \sum_{n\in\mathbb{N}} \frac{t^n}{n!}Q^n, \quad t \in \mathbb{R}_+. \tag{1}$$

This relation implies that the probability transition kernel can be written in terms of this fundamental generator matrix Q. On first glance, this relation doesn't provide much insight into the characteristics of the generator matrix. We will formally define generator matrix below, and relate this matrix to the jump transition probability matrix $p \in [0,1]^{\mathcal{X}\times\mathcal{X}}$ of the embedded Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, and transition rate sequence $\nu : \mathcal{X} \to \mathbb{R}_+$.

Definition 1.1 (Generator matrix). For a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with transition kernel function $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$, the **generator matrix** $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ is defined as the following limit when it exists

$$Q \triangleq \lim_{t\downarrow 0} \frac{P(t) - I}{t}.$$

Remark 1. From Eq. (1), it is clear that the generator matrix is the limit defined above.

Theorem 1.2. *For a homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, the generator matrix exists and is defined in terms of sojourn time transition rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$, and jump transition matrix $p \in [0,1]^{\mathcal{X}\times\mathcal{X}}$ as*

$$Q_{xx} = -\nu_x, \qquad\qquad Q_{xy} = \nu_x p_{xy}.$$

Proof. Consider a fixed time $t \in \mathbb{R}_+$ and states $x, y \in \mathcal{X}$. We can expand the (x,y)th entry of transition matrix in terms of disjoint events $\{N_t = n\}$ as

$$P_{xy}(t) = P_x\{X_t = y\} = \sum_{n\in\mathbb{Z}_+} P_x\{X_t = y, N_t = n\}.$$

We can write the upper and lower bound the transition probability $P_{xy}(t)$ as

$$\sum_{n=0}^{1} P_x\{X_t = y, N_t = n\} \leqslant P_{xy}(t) \leqslant \sum_{n=0}^{1} P_x\{X_t = y, N_t = n\} + P\{N_t \geqslant 2\}.$$

Since $I_{xy} = \mathbb{1}_{\{x\neq y\}}$, we can write the probabilities in terms of identity operator I as

$$P_x\{X_t = y, N_t = 0\} = I_{xy}e^{-\nu_x t}, \qquad P_x\{X_t = y, N_t = 1\} = (1 - I_{xy})p_{xy}\int_0^t \nu_x e^{-\nu_y(t-u)}e^{-\nu_x u}du.$$

The second equality follows from the nested conditional expectation. In particular, we have

$$P_x\{X_t = y, N_t = 1\} = \mathbb{1}_{\{x\neq y\}}\mathbb{E}_x\mathbb{1}_{\{X_t=y, S_1\leqslant t<S_2\}} = \mathbb{1}_{\{x\neq y\}}\mathbb{E}_x\mathbb{E}[\mathbb{1}_{\{X_t=y, T_2>t-S_1, S_1\leqslant t\}}|\mathcal{F}_{S_1}] = (1 - I_{xy})p_{xy}\mathbb{E}_x\mathbb{1}_{\{S_1\leqslant t\}}e^{-\nu_y(t-S_1)}.$$

Since $\{N_t \geqslant 2\}$ is of order $o(t)$ for small t, we can write

$$\frac{P_{xy}(t) - I_{xy}}{t} = -\nu_x I_{xy}\left(\frac{1 - e^{-\nu_x t}}{\nu_x t}\right) + \nu_x p_{xy}\frac{(e^{-\nu_y t} - e^{-\nu_x t})}{(\nu_x - \nu_y)t}(1 - I_{xy}) + o(t).$$

Taking limit as $t \downarrow 0$, we get the result. □

Corollary 1.3. *For each state $x \in \mathcal{X}$, the generator matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ for a pure-jump homogeneous CTMC satisfies*

$$0 \leqslant -Q_{xx} < \infty, \qquad 0 \leqslant Q_{xy} < \infty,\ y \in \mathcal{X} \qquad \sum_{y\in\mathcal{X}} Q_{xy} = 0.$$

Remark 2. From the semigroup property of probability kernel function and definition of generator matrix, we get the backward equation

$$\frac{dP(t)}{dt} = \lim_{s\downarrow 0} \frac{P(s+t)-P(t)}{s} = \lim_{s\downarrow 0} \frac{(P(s)-I)}{s} P(t) = QP(t), \quad t \in \mathbb{R}_+.$$

Similarly, we can also get the forward equation

$$\frac{dP(t)}{dt} = \lim_{s\downarrow 0} \frac{P(s+t)-P(t)}{s} = P(t) \lim_{s\downarrow 0} \frac{(P(s)-I)}{s} = P(t)Q, \quad t \in \mathbb{R}_+.$$

Both these results need a formal justification of exchange of limits and summation, and we next present a formal proof for these two equations.

Theorem 1.4 (backward equation). *For a homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with transition kernel function $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ and generator matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$, we have*

$$\frac{dP(t)}{dt} = QP(t), \quad t \in \mathbb{R}_+.$$

Proof. Fix states $x, y \in \mathcal{X}$ and we consider the lim inf and lim sup of (x,y)th term of $\frac{(P(s)-I)}{s}P(t)$. For any finite subset $F \subseteq \mathcal{X}$ containing x, we obtain

$$\liminf_{s\downarrow 0} \sum_{z\in\mathcal{X}} \frac{(P_{xz}(s)-I_{xz})}{s} P_{zy}(t) \geqslant \sum_{z\in F} \liminf_{s\downarrow 0} \frac{(P_{xz}(s)-I_{xz})}{s} P_{zy}(t) = \sum_{z\in F} Q_{xz}P_{zy}(t).$$

The above inequality holds for any finite set $F \subseteq \mathcal{X}$, and thus taking supremum over increasing sets F, we get the lower bound. For the upper bound, we observe for any finite subset $F \subseteq \mathcal{X}$ containing state x, we have

$$\limsup_{s\downarrow 0} \sum_{z\in\mathcal{X}} \frac{(P_{xz}(s)-I_{xz})}{s} P_{zy}(t) \leqslant \limsup_{s\downarrow 0} \left(\sum_{z\in F} \frac{(P_{xz}(s)-I_{xz})}{s} P_{zy}(t) - \sum_{z\in F} \frac{(P_{xz}(s)-I_{xz})}{s} \right) = \sum_{z\in F} Q_{xz}P_{zy}(t) - \sum_{z\in F} Q_{xz}.$$

The above inequality holds for any finite set $F \subseteq \mathcal{X}$, and thus taking infimum over increasing sets F and recognizing that $\sum_{z\in\mathcal{X}} Q_{xz} = 0$, we get the upper bound. ◻

Theorem 1.5 (forward equation). *For a homogeneous CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with transition kernel function $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ and generator matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$, we have*

$$\frac{dP(t)}{dt} = P(t)Q, \quad t \in \mathbb{R}_+.$$

Proof. Fix states $x, y \in \mathcal{X}$ and we consider the lim inf and lim sup of (x,y)th term of $P(t)\frac{(P(s)-I)}{s}$. We take a finite set $F \subseteq \mathcal{X}$ containing state y, to obtain the lower bound

$$\liminf_{s\downarrow 0} \sum_{z\in\mathcal{X}} P_{xz}(t) \frac{(P_{zy}(s)-I_{zy})}{s} \geqslant \sum_{z\in F} \liminf_{s\downarrow 0} P_{xz}(t) \frac{(P_{zy}(s)-I_{zy})}{s} \geqslant \sum_{z\in F} P_{xz}(t)Q_{zy}.$$

By taking limiting value for increasing sequence of finite sets $F \subseteq \mathcal{X}$, we obtain the lower bound. To obtain the upper bound, we observe for any finite subset $F \subseteq \mathcal{X}$ containing state y, we have

$$\limsup_{s\downarrow 0} \sum_{z\in\mathcal{X}} P_{xz}(t) \frac{(P_{zy}(s)-I_{zy})}{s} \leqslant \limsup_{s\downarrow 0} \left(\sum_{z\in F} P_{xz}(t) \frac{(P_{zy}(s)-I_{zy})}{s} + \sum_{z\notin F} \frac{P_{zy}(s)}{s} \right) = \sum_{z\in F} P_{xz}(t)Q_{zy} + \sum_{z\notin F} Q_{zy}.$$

The second equality follows from monotone convergence theorem. Taking infimum over increasing sets F and from the fact that $\sum_{y\in\mathcal{X}\setminus\{x\}} p_{xy} = 1$, we get the upper bound. ◻

Remark 3. Recall that for a homogeneous discrete time Markov chain with one-step transition probability matrix P, we can write the n-step transition probability matrix $P^{(n)} = P^n$. That is, for any given stochastic matrix P, we can construct a discrete time Marko chain. We can generalize this notion to homogeneous continuous time Markov chains as well. Given a matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ that satisfies the properties of a generator matrix given in Corollary 1.3, we can construct a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ by finding its transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$, by defining $P(t) \triangleq e^{tQ}$ for all $t \in \mathbb{R}_+$. We observe that $P(1) = e^Q$ and we have $P(t) = P(1)^t$ for all $t \in \mathbb{R}_+$. We need to show that such a defined function is indeed a probability transition kernel. We will first show that such a function P satisfies some of the properties of the transition kernel, and then show that $P(t)$ is transition matrix at all times $t \in \mathbb{R}_+$.

Theorem 1.6. *Let $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ be a matrix that satisfies the properties of generator matrix given in Corollary 1.3. We define a function $P : \mathbb{R}_+ \to \mathbb{R}_+^{\mathcal{X}\times\mathcal{X}}$ by $P(t) \triangleq e^{tQ}$ for all $t \in \mathbb{R}_+$. Then the function P satisfies the following properties.*

1. *P has the semigroup property, i.e. $P(s+t) = P(s)P(t)$ for all $s,t \in \mathbb{R}_+$.*
2. *P is the unique solution to the forward equation, $\frac{dP(t)}{dt} = P(t)Q$ with initial conditon $P(0) = I$.*
3. *P is the unique solution to the backward equation, $\frac{dP(t)}{dt} = QP(t)$ with initial condition $P(0) = I$.*
4. *For all $k \in \mathbb{N}$, we have $\left.\frac{d^kP(t)}{d^k(t)}\right|_{t=0} = Q^k$.*

Proof. Given the definition of P and properties of Q, one can easily check these properties. □

Theorem 1.7. *A finite matrix $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ satisfies the properties of a generator matrix given in Corollary 1.3 iff the function $P : \mathbb{R}_+ \to \mathbb{R}_+^{\mathcal{X}\times\mathcal{X}}$ defined by $P(t) \triangleq e^{tQ}$ is a stochastic matrix for all $t \in \mathbb{R}_+$.*

Proof. Sufficiency has already been seen before, and hence we will focus only on necessity. Accordingly, we assume that $Q \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ satisfies the properties of a generator matrix given in Corollary 1.3, then we will show that $P(t) = e^{tQ}$ is a stochastic matrix. Recall that $Q\mathbf{1}^T = 0$ for all ones column vector $\mathbf{1}^T$, and hence $Q^n\mathbf{1}^T = 0$ for all $n \in \mathbb{N}$. Expanding $P(t)$ in terms of expression for matrix exponentiation, we write $P(t) = I + \sum_{k\in\mathbb{N}} \frac{t^n}{n!}Q^n$. This implies that $P(t)\mathbf{1}^T = \mathbf{1}^T$. □

1.1 Transition graph

The weighted directed transition graph (V,E,w) consists of vertex set $V = \mathcal{X}$ and the edges being

$$E = \left\{(x,y) \in \mathcal{X}\times\mathcal{X} : Q_{xy} > 0, y \neq x\right\}.$$

The weights $w : E \to \mathbb{R}_+$ of the directed edges are given by $w_{xy} = Q_{xy} = \nu_x p_{xy}$.

2 Uniformization

Consider a homogeneous continuous-time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ in which the mean time spent in a state is identical for all states, i.e. $\nu_x = \nu$ uniformly for all states $x \in \mathcal{X}$. Recall that $N_t = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ denotes the number of state transitions until time $t \in \mathbb{R}_+$. Since the random amount of time spent in each state is *i.i.d.* with common exponential distribution of rate ν, the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a Poisson process with rate ν. In this case, we can explicitly characterize the transition kernel function $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ for this CTMC X in terms of the jump transition probability matrix $p \in [0,1]^{\mathcal{X}\times\mathcal{X}}$ and uniform transition rate ν. To this end, we use the law of total probability over countable partitions $(\{N_t = n\} : n \in \mathbb{Z}_+)$ to get

$$P_{xy}(t) = \sum_{n\in\mathbb{Z}_+} P_x\{N(t) = n\} P(\{X_t = y\} \mid \{X_0 = x, N_t = n\}) = \sum_{n\in\mathbb{Z}_+} p_{xy}^{(n)} e^{-\nu t}\frac{(\nu t)^n}{n!}.$$

This equation could also have been derived by observing that $Q = -\nu(I-p)$ and hence using the exponentiation of matrix, we can write

$$P(t) = e^{-\nu t(I-p)} = e^{-\nu t}e^{\nu t p} = e^{-\nu t}\sum_{n\in\mathbb{Z}_+} p^n\frac{(\nu t)^n}{n!}. \tag{2}$$

Eq. (2) gives a closed form expression for $P(t)$ and also suggests an approximate computation by an appropriate partial sum. However, its application is limited as the transition rates for all states are all assumed to be equal. It turns out that any regular Markov chain can be transformed in this form by allowing hypothetical transitions from a state to itself.

2.1 Uniformization step

Consider a regular CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with bounded transition rates, with finite rate ν such that $\nu_x \leqslant \nu$ for all states $x \in \mathcal{X}$. Since from each state $x \in \mathcal{X}$, the Markov chain leaves at rate ν_x, we could equivalently assume that the transitions occur at a rate ν but only $\frac{\nu_x}{\nu}$ are real transitions and the remaining transitions are fictitious self-transitions.

Construction 2.1 (uniformization). For any regular continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with transition rate $\nu : \mathcal{X} \to \mathbb{R}_+$ and jump probability transition matrix $p \in [0,1]^{\mathcal{X}\times\mathcal{X}}$, we can find a finite rate $\nu \geqslant \sup_{x\in\mathcal{X}} \nu_x$. We construct a continuous time Markov chain $Y : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with uniform transition rates ν for all states $x \in \mathcal{X}$, and jump probability transition matrix $q \in [0,1]^{\mathcal{X}\times\mathcal{X}}$ defined as

$$q_{xy} = \frac{\nu_x}{\nu} p_{xy} \mathbb{1}_{\{y\neq x\}} + \left(1 - \frac{\nu_x}{\nu}\right) \mathbb{1}_{\{y=x\}}, \quad x, y \in \mathcal{X}.$$

The process Y is called the **uniformized** version of process X. This technique of uniformizing the rate in which a transition occurs from each state to any other state by introducing self transitions is called **uniformization**.

Theorem 2.2. *A regular CTMC X and its uniformized version Y are identical in distribution.*

Proof. We consider the *i.i.d.* sequence of transition times $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with the common exponential distribution of rate ν for the Markov process Y. Assuming the initial state x for the Markov process Y, we define a random sequence of indicators $\xi : \Omega \to \{0,1\}^{\mathbb{N}}$, defined by

$$\xi_n \triangleq \mathbb{1}_{\{Y_{T_n} \neq x\}}, \quad n \in \mathbb{N}.$$

From the definition of uniformized process Y, we know that $P_x\{\xi_1 = \xi_2 = \cdots = \xi_n = 0\} = q_{xx}^n = (1 - \frac{\nu_x}{\nu})^n$, and ξ is an *i.i.d.* sequence. We can define the corresponding counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ that counts the number of transitions to exit state x, as

$$N \triangleq \inf\{n \in \mathbb{N} : \xi_n = 1\}.$$

Since ξ is *i.i.d.* Bernoulli, N is a geometric random variable with success probability $1 - q_{xx} = \frac{\nu_x}{\nu}$. To show that the two Markov processes Y and X have identical distribution, it suffices to show that

(a) $U \triangleq \sum_{n=1}^{N} T_n$ is distributed exponentially with rate ν_x, and

(b) $P(\{Y_U = y\} \mid \{Y_0 = x\}) = p_{xy}$.

To see (a), we observe that random sequence T and random variable N are independent, and hence we can compute the moment generating function of U as

$$M_U(\theta) = \mathbb{E}\left[\mathbb{E}[\prod_{n=1}^{N} e^{-\theta T_n} | N]\right] = \mathbb{E} M_{T_1}^N(\theta) = \sum_{n\in\mathbb{N}} \left(\frac{\nu}{\nu+\theta}\right)^n q_{xx}^{n-1}(1-q_{xx}) = \frac{\nu_x}{\nu_x+\theta}.$$

To see (b), from the Markov property of process Y and its embedded jump transition matrix q, we observe that

$$P_x\{Y_U = y\} = \sum_{n\in\mathbb{N}} P_x\{Y_U = y, N = n\} = \sum_{n\in\mathbb{N}} P_x\{Y_1 = \cdots = Y_{n-1} = x, Y_n = y\} = \sum_{n\in\mathbb{N}} q_{xy} q_{xx}^{n-1} = \frac{q_{xy}}{1-q_{xx}} = p_{xy}.$$

□

Remark 4. Any regular continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ can be thought of as being in a process that spends a random time in state $x \in \mathcal{X}$ distributed exponentially with rate ν, and then makes a transition to state $y \in \mathcal{X}$ with probability p_{xy}^*. Then, one can write the probability transition kernel as

$$P_{xy}(t) = \sum_{n=0}^{\infty} q_{xy}^{(n)} e^{-\nu t} \frac{(\nu t)^n}{n!}.$$

3 Class Properties

Definition 3.1. For a CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ defined on the countable state space $\mathcal{X} \subseteq \mathbb{R}$, we say a state y is **reachable** from state x if $P_{xy}(t) > 0$ for some $t > 0$, and we denote $x \to y$. If two states $x, y \in \mathcal{X}$ are reachable from each other, we say that they **communicate** and denote it by $x \leftrightarrow y$.

Lemma 3.2. *Communication is an equivalence relation.*

Definition 3.3. Communication equivalence relation partitions the state space $\mathcal{X}$ into equivalence classes called **communicating classes**. A CTMC with a single communicating class is called **irreducible**.

Theorem 3.4. *A regular CTMC and its embedded DTMC have the same communicating classes.*

Proof. It suffices to show that $x \to y$ for the regular Markov process iff $x \to y$ in the embedded chain. If $x \to y$ for embedded chain, then there exists a path $x = x_0, x_1, \dots, x_n = y$ such that $p_{x_0 x_1} p_{x_1 x_2} \cdots p_{x_{n-1} x_n} > 0$ and $\nu_{x_0} \nu_{x_1} \dots \nu_{x_{n-1}} > 0$. It follows that S_n is a proper random variable, and we can write

$$P_{xy}(t) \geqslant \prod_{k=0}^{n-1} p_{x_k x_{k+1}} \mathbb{E}[P\{T_{n+1} > t - S_n\}] > 0.$$

Conversely, if the states y is not reachable from state x in embedded chain, then it won't be reachable in the regular CTMC. □

Corollary 3.5. *A regular CTMC is irreducible iff its embedded DTMC is irreducible.*

Remark 5. There is no notion of periodicity in CTMCs since there is no fundamental time-step that can be used as a reference to define such a notion. In fact, for any state $x \in \mathcal{X}$ of a non-instantaneous homogeneous CTMC we have $P_{xx}(t) > e^{-\nu_x t} > 0$ for all $t \geqslant 0$.

3.1 Recurrence and transience

Definition 3.6. For any state $y \in \mathcal{X}$, we define the first hitting time to state y after leaving state y as

$$\tau_y^+ = \inf\{t > Y_0 : X_t = y\}.$$

The state y is said to be **recurrent** if $P_y\left\{\tau_y^+ < \infty\right\} = 1$ and **transient** if $P_y\left\{\tau_y^+ < \infty\right\} < 1$. Furthermore, a recurrent state y is said to be **positive recurrent** if $\mathbb{E}_y \tau_y^+ < \infty$ and **null recurrent** if $\mathbb{E}_y \tau_y^+ = \infty$.

Theorem 3.7. *An irreducible CTMC is recurrent iff its embedded DTMC is recurrent.*

Proof. There is nothing to prove for $|\mathcal{X}| = 1$. Hence, we assume $|\mathcal{X}| \geqslant 2$ without loss of generality. Suppose that the embedded Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ is recurrent. Let the initial state $Z_0 = x \in \mathcal{X}$, the number of visits to state y during successive visit to state x be denoted by N_{xy}, and the kth sojourn time in state y by $Y_k^{(y)}$. Since the embedded chain is irreducible, it has no absorbing states. This implies N_{xy} and $\sum_{y \in \mathcal{X}} N_{xy}$ are finite almost surely, and the random sequence $Y^{(y)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* exponential with rate $\nu_y \in (0, \infty)$, and sequences $Y^{(y)}$ are independent for each state $y \in \mathcal{X}$. Since we can write $\tau_x^+ = \sum_{y \in \mathcal{X}} \sum_{k=1}^{N_{xy}} Y_k^{(y)}$, it follows that the recurrence time τ_x^+ is finite almost surely.

Conversely, if the embedded Markov chain is not recurrent, it has a transient state $x \in \mathcal{X}$ for which $P_x\{N_x = \infty\} > 0$. By the same argument, $P_x\{\tau_x^+ = \infty\} > 0$ and hence the CTMC is not recurrent. □

Remark 6. An irreducible regular CTMC maybe null recurrent where embedded Markov chain is positive recurrent.

Corollary 3.8. *Recurrence is a class property.*

Theorem 3.9. *Consider an irreducible recurrent CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and transition matrix $p \in [0,1]^{\mathcal{X} \times \mathcal{X}}$ for the embedded Markov chain. Let $u \in \mathbb{R}_+^{\mathcal{X}}$ be any strictly positive solution of $u = up$, then*

$$\mathbb{E}_x \tau_x^+ = \frac{1}{u_x} \sum_{y \in \mathcal{X}} \frac{u_y}{\nu_y}, \quad x \in \mathcal{X}.$$

Proof. Let $X_0 = x \in \mathcal{X}$, and N_{xy} be the number of visits to state $y \in \mathcal{X}$ between successive visits to state x in the embedded Markov chain. From the recurrence of the embedded Markov chain, we know that for any strictly positive solution to $u = uP$ we have $\mathbb{E}_x N_{xy} = \frac{u_y}{u_x}$. Let $Y_k^{(x)}$ denote the sojourn time of the CTMC X in state x during the kth visit. The random sequence $Y^{(x)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* exponential with rate ν_x. Therefore, we can write

$$\tau_x^+ = Y_0^x + \sum_{y \in \mathcal{X} \setminus \{x\}} \sum_{k=1}^{N_{xy}} Y_k^{(y)}.$$

We recall that jump chain and sojourn times are independent given the initial state, and hence N_{xy} and $Y^{(y)}$ sequences are independent for each state $y \neq x$. Result follows from taking expectations on both sides, exchanging summation and expectations for positive random variables, to get

$$\mathbb{E}_x \tau_x^+ = \mathbb{E}_x Y_0^x + \sum_{y \in \mathcal{X} \setminus \{x\}} \mathbb{E}_x \sum_{k=1}^{N_{xy}} Y_k^{(y)} = \sum_{y \in \mathcal{X}} \mathbb{E} Y_k^{(y)} \mathbb{E}_x N_{xy}.$$

□

Corollary 3.10. *Consider an irreducible recurrent CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and the transition matrix p for the embedded Markov chain. Let u be any strictly positive solution to $u = up$. Then, CTMC X is positive recurrent iff $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} < \infty$. In particular, the CTMC is positive recurrent iff $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} = 1$.*

4 Stationarity

Definition 4.1. A map $\pi : \mathcal{X} \to [0,1]$ is an **equilibrium distribution** of a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with probability transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X} \times \mathcal{X}}$ if $\mathbf{1}\pi = 1$ and $\pi P(t) = \pi$ for all $t \in \mathbb{R}_+$.

Remark 7. Let $\pi(0)$ denote the marginal distribution of initial state X_0, then by definition of probability transition kernel for Markov process X, we can write the marginal distribution of X_t as

$$\pi(t) = \pi(0)P(t), \quad t \in \mathbb{R}_+.$$

In general, we can write $\pi(s+t) = \pi(s)P(t)$, and hence if there exists a stationary distribution $\pi \triangleq \lim_{s \to \infty} \pi(s)$ for this process X, then we would have $\pi = \pi P(t)$ for all times $t \in \mathbb{R}_+$.

Remark 8. Recall that an irreducible DTMC is positive recurrent iff it has a strictly positive stationary distribution.

Corollary 4.2. *For a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with generator matrix Q, a distribution $\pi : \mathcal{X} \to [0,1]$ is an equilibrium distribution iff $\pi Q = 0$.*

Proof. Recall that we can write the transition probability matrix $P(t)$ at any time $t \in \mathbb{R}_+$ in terms of generator matrix Q as $P(t) = e^{tQ}$. Using the exponentiation of a matrix, we can write

$$\pi P(t) = e^{tQ} = \pi + \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \pi Q^n, \quad t \in \mathbb{R}_+.$$

Therefore, $\pi Q = 0$ iff π is an equilibrium distribution of the Markov process X. □

Theorem 4.3. *Let $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ be an irreducible recurrent homogeneous CTMC with probability transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X} \times \mathcal{X}}$, the transition rate sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, and the transition matrix for embedded jump chain $p \in [0,1]^{\mathcal{X}}$. Then for all states $x, y \in \mathcal{X}$ the $\lim_{t \to \infty} P_{xy}(t)$ exists, this limit is independent of the initial state $x \in \mathcal{X}$ and denoted by π_y. Let u be any strictly positive invariant measure such that $u = up$. If $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} = \infty$, then $\pi_x = 0$ for all $x \in \mathcal{X}$. If $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} < \infty$ then for all $y \in \mathcal{X}$,*

$$\pi_y = \frac{\frac{u_y}{\nu_y}}{\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x}} = \frac{\nu_y^{-1}}{\mathbb{E}_y \tau_y^+}.$$

Proof. Fix a state $y \in \mathcal{X}$, and define a process $W : \Omega \to \{0,1\}^{\mathbb{R}_+}$ such that $W_t = \mathbb{1}_{\{X_t = y\}}$. Then, from the regenerative property of the homogeneous CTMC and renewal reward theorem, we have

$$\lim_{t \to \infty} P_x \{X_t = y\} = \frac{\nu_y^{-1}}{\mathbb{E}_y \tau_y^+}.$$

□