Lecture-23: Queueing Networks

1 Migration Processes

Corollary 1.1. *Consider an M/M/s queue with Poisson arrivals of rate λ and each server having exponential service of rate μ. If $\lambda < s\mu$, then the output process in steady state is also Poisson with rate λ.*

Proof. Let X_t denote the number of customers in the system at time t. Since M/M/s process is a birth and death process, it follows from the previous proposition that $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}}$ is time reversible for X at stationarity. The stationarity of the forward process holds only if the queue is stable, i.e. $\lambda < s\mu$.

Now going forward in time, the time instants at which X_t increases by unity are the arrival instants of a Poisson process. Hence, by time reversibility, the time points at which X_t increases by unity when we go backwards in time also constitutes a Poisson process. But these instants are exactly the departure instants of the forward process. Hence, the result holds when $\lambda < s\mu$. □

Lemma 1.2. *For an ergodic M/M/1 queue in steady state, the following are true.*

1. *The number of customers present in the system at time t is independent of the sequence of past departures.*
2. *For FCFS service discipline, the waiting time spent in the system (waiting in the queue plus the service time) by a customer is independent of the departure process prior to its departure.*

Proof. Proofs follow by looking at reversed process.

1. Since the arrival process is Poisson, the future arrivals are independent of the number of customers in the system at the current instant. Looking backwards in time, future arrivals are the past departures. Hence by time reversibility, the number of customers currently in the system are also independent of the past departures.
2. Consider the case when a customer arrives into the system at time T_1. The customer leaves at time $T_2 > T_1$. Since the service discipline is assumed first come first serve and the arrival is Poisson, it is seen that the waiting time $T_2 - T_1$ is independent of the customers coming after T_1. Looking at the reversed process, we see that $T_2 - T_1$ will be independent of the arrivals after T_2 for the reversed process.

□

2 Network of Queues

2.1 Tandem Queues

Time reversibility of M/M/s queues can be used to study what is called as a tandem or sequential queueing system. For instance, consider a queueing system wih two queues in sequence, with each queue having one dedicated server. Service time of server i is random independent and distributed exponentially with rate μ_i. Customers arrive to the queue 1 according to a Poisson process with rate λ. After being served by server 1, customers join queue 2 for its service. Assume there is infinite waiting room at both servers. Since the departure process of queue 1 is Poisson, as discussed previously, the arrival process to queue 2 is also Poisson with rate λ. Time reversibility concept can be used to give a much stronger result.

Theorem 2.1. *For the ergodic tandem queue in steady state, the following are true.*

1. *The number of customers X_1, X_2 present at server 1 and server 2 respectively, are independent, and*

$$P\{X_1 = n_1, X_2 = n_2)\} = {\rho_1}^{n_1}(1-\rho_1){\rho_2}^{n_2}(1-\rho_2).$$

2. *For FCFS discipline, the waiting time at server 1 is independent of the waiting time at server 2.*

Proof. Proofs follow by looking at reversed process.

1. By part 1 of previous lemma, we have that the number of customers at server 1 is independent of the past departures of server 1. But past departures are same as the arrival to server 2. Thus follows the independence of the number of customers in both servers. The formula for the joint density follows from the independence and the formula for the limiting probabilities of an M/M/1 queue.

2. By part 2 of the previous lemma, the waiting time of a customer at server 1 is independent of the past departures happening at server 1. But the past departures at server 1, in conjunction with the service times at server 2, determine customer's waiting time at server 2. Hence the result follows.

□

2.2 Jackson Network

Consider a system of k queues each with a dedicated server with random independent service times distributed exponentially with rate μ_i for server $i \in [k]$. We assume an infinite waiting time at each of the k queues. To each queue, customers arrive from outside the system, according to a Poisson process with rate r_i. Once a customer is served at server i, the customer joins queue j with probability P_{ij}, such that $\sum_{j\in[k]} P_{ij} \leqslant 1$. The probability of the customer departing the system is $1 - \sum_{j\in[k]} P_{ij}$. If we denote λ_j as the total rate at which the customers join queue j, then λ_j can be obtained as a solution to

$$\lambda_j = r_j + \sum_{i\in[k]} \lambda_i P_{ij}, \quad j \in [k].$$

Denoting the the number of customers in server $i \in [k]$ at time $t \in \mathbb{R}$ by $X_i(t) \in \mathbb{Z}_+$, we can analyze this model by a stationary continuous-time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ with $\mathcal{X} = \mathbb{Z}_+^k$ where the process at any time $t \in \mathbb{R}$ is denoted by $X(t) = (X_1(t), X_2(t), \dots X_k(t)) \in \mathbb{Z}_+^k$. From the tandem queue results, we expect the customers at each server to be independent random variables. Let $n \in \mathbb{Z}_+^k$, then we are interested in knowing the

$$\pi(n) \triangleq P\{X(t) = n\} = P\{X_1(t) = n_1, \dots, X_k(t) = n_k\}.$$

We write the marginal probabilities as

$$\pi_i(m_i) \triangleq P\{X_i(t) = m_i\} = \sum_{n\in\mathbb{Z}_+^k : n_i = m_i} \pi(n).$$

Since, X is a CTMC, only a single transition takes place in any infinitesimal time interval. Let $X(t) = n \in \mathbb{Z}_+^k$, then the possible transitions from this Markov process at time $t \in \mathbb{R}$ and the associated rates are the following. We denote the unit vector in the i direction by e_i.

i_ An external arrival takes place in queue i, with rate

$$Q(n, n + e_i) = r_i.$$

ii_ If $n_i > 0$, a service completes and a customer departs from queue i exiting the system, with rate

$$Q(n, n - e_i) = \mu_i(1 - \sum_{j\in[k]} P_{ij}).$$

iii_ If $n_i > 0$, a service completes and a customer departs from queue i and joins queue j, with rate

$$Q(n, n - e_i + e_j) = \mu_i P_{ij}.$$

Theorem 2.2. *If $\lambda_i < \mu_i$ for each $i \in [k]$, then the reversed stochastic process is a network process of the same type as original. That is, the reversed process $\hat{X}$ is a CTMC with the generator matrix Q^* given by the following.*

(i) The system has external Poisson arrivals to queue i at rate $\lambda_i(1 - \sum_{j\in[k]} P_{ij})$. That is,

$$Q^*(n, n + e_i) = \lambda_i(1 - \sum_{j\in[k]} P_{ij}), \qquad i \in [k].$$

(ii) The service times at queue i is i.i.d. *exponential with rate μ_i.*

(iii) A departure from queue j goes to queue i with probability $\bar{P}_{ji}$ as given by

$$\bar{P}_{ji} = \frac{\lambda_i P_{ij}}{\lambda_j}.$$

That is, if $n_j > 0$, then a customer joins queue i after a service completion from queue j with rate

$$Q^*(n, n - e_j + e_i) = \mu_j \bar{P}_{ji} = \frac{\mu_j}{\lambda_j} \lambda_i P_{ij}.$$

The corresponding rate of customer departing the system after service completion from server j is

$$Q^*(n, n - e_j) = \mu_j (1 - \sum_{i \in [k]} \bar{P}_{ji}) = \frac{\mu_j}{\lambda_j} (\lambda_j - \sum_{i \in [k]} \lambda_i P_{ij}) = \frac{\mu_j}{\lambda_j} r_j.$$

(iv) For $\rho_i \triangleq \frac{\lambda_i}{\mu_i}$, the stationary distribution satisfies

$$\pi(n) = \prod_{i=1}^{k} \pi_i(n_i) = \prod_{i=1}^{k} \rho_i^{n_i}(1 - \rho_i), \qquad n \in \mathbb{Z}_+^k.$$

Proof. It suffices to check that the detailed balanced conditions hold with the candidate generator matrix Q^* for the reversed process and the candidate distribution π.

We first focus at the detailed balance equations associated with the external arrivals

$$\pi(n) Q(n, n + e_i) = \pi(n + e_i) Q^*(n + e_i, n), \qquad n \in \mathbb{Z}_+.$$

Since $Q(n, n + e_i) = r_i, Q^*(n + e_i, n) = \frac{r_i}{\rho_i}$ and the candidate equilibrium has a product form, we have

$$\pi_i(n_i + 1) = \rho_i \pi_i(n_i).$$

That the detailed balance equations hold for these transitions.

Second for $n_i > 0$, we look at the detailed balanced equations corresponding the exits from the system from queue i,

$$\pi(n) Q(n, n - e_i) = \pi(n - e_i) Q^*(n - e_i, n).$$

From the product form for equilibrium distribution π, we have for each $n_i \in \mathbb{N}$

$$\pi_i(n_i) \mu_i (1 - \sum_{j \in [k]} P_{ij}) = \pi_i(n_i - 1) \lambda_i (1 - \sum_{j \in [k]} P_{ij}).$$

From the candidate equilibrium distribution π, the detailed balance equations continue to hold for these transitions as well.

Finally for $n_i > 0$, we look at the detailed balanced equations corresponding the transitions where a customer joins queue j after getting service from queue i,

$$\pi(n) Q(n, n - e_i + e_j) = \pi(n - e_i + e_j) Q^*(n - e_i + e_j, n).$$

From the product form of the candidate equilibrium distribution π and the definition of Q and candidate construction Q^*, we get that

$$\pi_i(n_i) \pi_j(n_j) \mu_i P_{ij} = \pi_i(n_i - 1) \pi_j(n_j + 1) \mu_j \bar{P}_{ji}, \quad n_i \in \mathbb{N}.$$

From the candidate equilibrium distribution π and definition of $\bar{P}$, the detailed balance equations continue to hold for these transitions as well. □

Corollary 2.3. *The process of customers departing the system from the server i, $i = 1, 2 \ldots k$, are independent Poisson processes having respective rates $\lambda_i (1 - \sum_j P_{ij})$.*

Proof. We have already shown that in the reverse process, customers arrive to server i from outside the system according to independent Poisson processes having rates $\lambda_i (1 - \sum_{j \in [k]} P_{ij})$ for $i \in [k]$. Since an arrival from outside corresponds to a departure out of the system from server i in the forward process, the result follows. □