Lecture-28: Random Walks

1 Introduction

Definition 1.1. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a step-size sequence of *i.i.d.* random variables, where $\mathcal{X} \subseteq \mathbb{R}$ and $\mathbb{E}|X_n| < \infty$. We define $S_0 \triangleq 0$ and the location of a particle after n steps as $S_n \triangleq \sum_{i=1}^{n} X_i$. Then the sequence $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ is called a *random walk process*.

Example 1.2 (Simple random walk). If the step-size alphabet $\mathcal{X} = \{-1, 1\}$, then the random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ is **simple**.

Remark 1. Random walks are generalizations of renewal processes. If X was a sequence of non-negative random variables indicating inter-renewal times, then S_n is the instant of the nth renewal event.

2 Duality in random walks

Lemma 2.1 (Duality principle). *For any finite $n \in \mathbb{N}$, the joint distributions of finite sequence $(X_1, X_2, \cdots, X_n)$ and the reversed sequence $(X_n, X_{n-1}, \cdots, X_1)$ are identical, for any* i.i.d. *step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$.*

Proof. Since $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a sequence of *i.i.d.* random variables, it is exchangeable. The reversed sequence is $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ where $\sigma : [n] \to [n]$ is permutation with $\sigma(i) = n - i + 1$. □

Corollary 2.2. *For any random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$, the distributions of S_k and $S_n - S_{n-k}$ are identical for any $k \in [n]$.*

Proof. Using duality principle, we can write the following equality for any $x \in \mathbb{R}$ and step $k \in [n]$

$$P\{S_k \leqslant x\} = P\left\{\sum_{i=1}^{k} X_i \leqslant x\right\} = P\left\{\sum_{i=1}^{k} X_{n-i+1} \leqslant x\right\} = P\left\{\sum_{i=n-k+1}^{n} X_i \leqslant x\right\} = P\{S_n - S_{n-k} \leqslant x\}.$$

□

Corollary 2.3. *For any random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$, for any finite $n \in \mathbb{N}$, the joint distributions of finite sequence $(S_1, S_2, \cdots, S_n) \overset{d}{=} (S_n - S_{n-1}, S_n - S_{n-2}, \cdots, S_n)$.*

Proposition 2.4. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having positive mean. The first hitting time of the random walk S to set of positive real numbers, $\tau \triangleq \min\{n \in \mathbb{N} : S_n > 0\}$, has finite mean. That is, $\mathbb{E}\tau < \infty$.*

Proof. Consider a discrete process $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, where $T_0 \triangleq 0$ and for each $k \in \mathbb{Z}_+$

$$T_{k+1} \triangleq \inf\{n > T_k : S_n \leqslant S_{T_k}\} = T_k + \inf\{n \in \mathbb{N} : S_{T_k+n} \leqslant S_{T_k}\}.$$

We observe that T_k is a stopping time adapted to the natural filtration of step-size sequence X for each $k \in \mathbb{N}$. Further, we can write the difference $T_{k+1} - T_k = \inf\{n \in \mathbb{N} : \sum_{i=1}^{n} X_{T_k+i} \leqslant 0\}$, From the strong Markov property for *i.i.d.* sequences, the distribution of $(X_1, \dots, X_n)$ is identical to that of $(X_{T_k+1}, \dots, X_{T_k+n})$, for any finite $n \in \mathbb{N}$. Therefore, it follows that $S_{T_k+n} - S_{T_k}$ has identical distribution to S_n, and is independent of step-size process X stopped at time T_k. Hence, the sequence $(T_k - T_{k-1} : k \in \mathbb{N})$ is *i.i.d.* , with complementary distribution

$$\bar{F}(m) = P\{T_{k+1} - T_k > m\} = P\{T_1 > m\} = P\{S_1 > 0, S_2 > 0, \dots, S_m > 0\}.$$

Therefore, $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ is a renewal process such that $\{T_1 = n\}$ implies that $\{S_n \leqslant \min\{0, S_1, \dots, S_{n-1}\}\}$. That is, we can write

$$\{T_1 = n\} = \{S_1 > 0, \dots, S_{n-1} > 0\} \cap \{S_n \leqslant \min\{0, S_1, \dots, S_{n-1}\}\} = \{S_1 > 0, \dots, S_{n-1} > 0, S_n \leqslant 0\}.$$

Hence, T_k denotes the kth renewal instant corresponding to the random walk S_n hitting kth low. We can define the inverse counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ for this renewal process as $N_n \triangleq \sum_{j=1}^{n} \mathbb{1}_{\{T_j \leqslant n\}}$, or $\{N_n \geqslant k\} = \{T_k \leqslant n\}$. From definition of stopping time τ and duality principle, we can write

$$P\{\tau > n\} = P(\cap_{k=1}^{n} \{S_k \leqslant 0\}) = P(\cap_{k=1}^{n} \{S_n \leqslant S_{n-k}\}) = P\{S_n \leqslant \min\{0, S_1, \dots, S_{n-1}\}\} = P(\cup_{k=1}^{n} \{T_k = n\}).$$

The event of renewal process hitting a new low at n is same as some renewal occurring at time n. That is,

$$N_\infty = \sum_{k \in \mathbb{N}} \mathbb{1}_{\{T_k < \infty\}} = \sum_{k \in \mathbb{N}} \sum_{n \geqslant k} \mathbb{1}_{\{T_k = n\}} = \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} \mathbb{1}_{\{T_k = n\}}.$$

Therefore, we can write the mean of stopping time τ as

$$\mathbb{E}\tau = 1 + \sum_{n \in \mathbb{N}} P\{\tau > n\} = 1 + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} P\{T_k = n\} = 1 + \mathbb{E}N_\infty.$$

Since $\mathbb{E}X_1 > 0$, it follows from strong law of large numbers that $S_n \to \infty$. Hence, the expected number of renewals that occur is finite. Elaborate. Thus $\mathbb{E}N_\infty < \infty$ and hence $\mathbb{E}\tau < \infty$. □

Definition 2.5. Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with $S_0 \triangleq 0$. The number of distinct values of $(S_0, \cdots, S_n)$ is called **range**, denoted by R_n. We define the first hitting time of random walk S to $x \in \mathbb{R}$ as the stopping time

$$T_x \triangleq \inf\{n \in \mathbb{N} : S_n = x\}.$$

Proposition 2.6. *For a simple random walk,* $\lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n} = P\{T_0 = \infty\}$.

Proof. We can define indicator function for S_k being a distinct number from $S_0, \dots, S_{k-1}$, as

$$I_k \triangleq \mathbb{1}_{\{S_k \neq S_{k-1}, \dots, S_k \neq S_0\}}.$$

Then, we can write range R_n in terms of indicator I_k as $R_n = 1 + \sum_{k=1}^{n} I_k$. From the duality principle

$$P(\cap_{i=1}^{k} \{S_k \neq S_{k-i}\}) = P(\cap_{i=1}^{k} \{S_i \neq 0\}), \quad k \in \mathbb{N}.$$

Therefore, we can write

$$\mathbb{E}R_n = 1 + \sum_{k=1}^{n} P\{S_1 \neq 0, \dots, S_k \neq 0\} = \sum_{k=0}^{n} P\{T_0 > k\}.$$

Result follows by dividing both sides by n and taking limits. □

2.1 Simple random walk

Theorem 2.7 (range). *For a simple random walk with* $P\{X_1 = 1\} = p$, *the following holds*

$$\lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n} = \begin{cases} 2p - 1, & p > \frac{1}{2} \\ 2(1-p) - 1, & p \leqslant \frac{1}{2}. \end{cases}$$

Proof. When $p = \frac{1}{2}$, this random walk is recurrent and thus from the Proposition 2.5, we have

$$P\{T_0 = \infty\} = 0 = \lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n}.$$

For $p > \frac{1}{2}$, let $\alpha \triangleq P(\{T_0 < \infty\} \mid \{X_1 = 1\})$. Since $\mathbb{E}X > 0$, we know that $S_n \to \infty$ and hence

$$P(\{T_0 < \infty\} \mid \{X_1 = -1\}) = 1.$$

We can write unconditioned probability of return of random walk to 0 as

$$P\{T_0 < \infty\} = \alpha p + (1-p).$$

Since $T_0 = 2$ when $S_2 = 0$, we have $P(\{T_0 < \infty\} \mid \{S_2 = 0\}) = 1$. Conditioning on X_2, from strong law of large numbers, we get

$$\alpha = P(\{T_0 < \infty, X_2 = 1\} \mid \{S_1 = 1\}) + P(\{T_0 < \infty, X_2 = -1\} \mid \{S_1 = 1\}) = pP(\{T_0 < \infty\} \mid \{S_2 = 2\}) + (1-p).$$

From Markov property and homogeneity of random walk process, it follows that

$$P(\{T_0 < \infty\} \mid \{S_2 = 2\}) = \frac{P(T_0 < \infty, S_2 = 2)}{P(S_2 = 2))} = \frac{P(T_0 < \infty, T_1 < \infty, S_2 = 2)}{P(S_2 = 2))}$$
$$= P(\{T_0 < \infty\} \mid \{T_1 < \infty\})P(\{T_1 < \infty\} \mid \{S_2 = 2\}) = \alpha^2.$$

Elaborate this. We conclude $\alpha = \alpha^2 p + 1 - p$, and since $\alpha < 1$ due to transience, we get $\alpha = \frac{1-p}{p}$, and hence the result follows. We can show similarly for the case when $p < 1/2$. □