# Differential Encoding for Real-Time Status Updates


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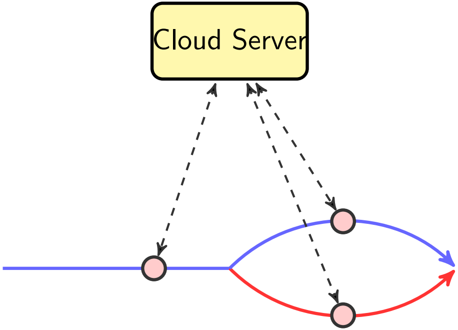
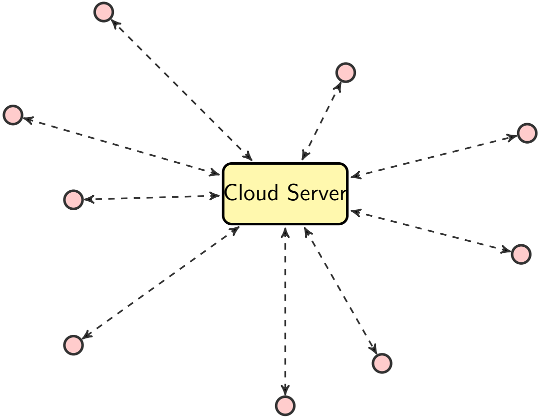
# Why timely update?



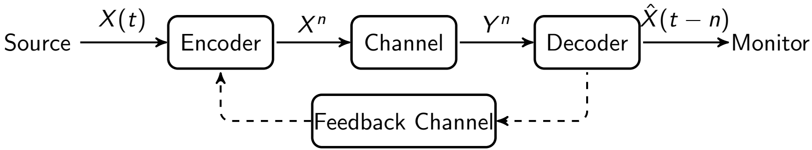

- Critical to know the status update before decision making

# Potential Scenarios




- Cyber-physical systems: Environmental/health monitoring
- Internet of Things: Real-time actuation/control

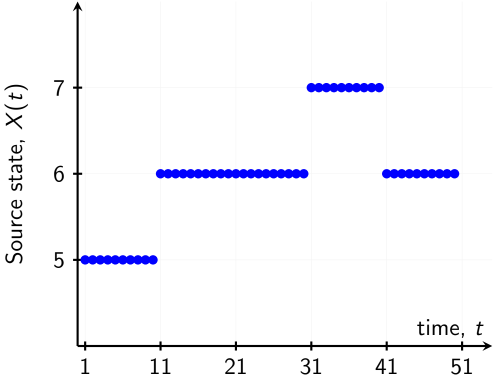
# Link Model




## Context

- Point-to-point communication with limited to no feedback
- Reliability through finite block-length coding

# Source Model




- Source state $X(t)$ can be represented by $m$ bits
- State difference between $n$ realizations can be represented by $k < m$ bits

# Problem Statement

### Question

How to encode message at the temporally correlated source for timely update? Should one send the current state or the difference between the current and the past state?

### Answer

It depends on the feedback

# Coding Model

- Finite length code of $n$ bits with permutation invariant code

## Updates

- True Update: current state $X(t)$ of $m$ bits is encoded to $n$ bit codeword $X^n$
- Incremental Update: the state difference $X(t) - X(t-n)$ of $k$ bits encoded to $n$ bit codeword $X^n$

# Channel Model

- Each transmitted bit of the codeword $X^n$ erased *iid* with probability $\epsilon$

### Erasure Distribution

Number of erasures is Binomial with parameter $(n, \epsilon)$

# Decoding and Reception

## Receiver Timing

Reception at time $t + n$ of $n$ bits sent at time $t$ after $n$ channel uses

## Probability of Decoding Failure

- True updates: $p_1 = \mathbb{E}P(n, n - m, E)$
- Incremental updates: $p_2 = EP(n, n - k, E)$
- Monotonicity: $0 < p_2 < p_1 < 1$

# Performance Metric

- Last successfully decoded source state at time $t$ was generated at $U(t)$
- *Information age*$A(t)$ at time $t$ as

$$A(t) = t - U(t).$$

- Limiting value of average age

$$\lim_{t \in \mathbb{R}} \frac{1}{t} \sum_{s=1}^{t} A(s).$$

# Update Transmission Schemes

## True Updates

- Each opportunity send *true update*

## Incremental Updates without Feedback

- Periodically send the *true update* after $q$ updates
- In between true updates, send *incremental updates*.

## Incremental Updates with Feedback

- Send the *true update* after each decoding failure
- In between true updates, send *incremental updates*.

# Renewal Reward Theorem: I

- Let $N(t)$ be a counting process with event instants $S_i$ for $i$th event
- Inter-renewal times $T_i = S_i - S_{i-1}$ are *iid*
- Reward in $i$th renewal interval $R_i$ is *iid*

## Renewal Reward Theorem

For cumulative reward process $R(t) = \sum_{i=1}^{N(t)} R_i$, the limiting average reward

$$\mathbb{E}R = \lim_{t \in \mathbb{R}} \frac{R(t)}{t} = \frac{\mathbb{E}R_i}{\mathbb{E}T_i}$$

# Renewal Reward Theorem: II

- Time instant $S_i$ of the $i$th successful reception of the true update
- For all three schemes, the $i$th inter-renewal time $T_i = S_i - S_{i-1}$ is *iid*
- Associated counting process $N(t)$ is a renewal process

# Renewal Reward Theorem: III

- Accumulated age in $i$th renewal period

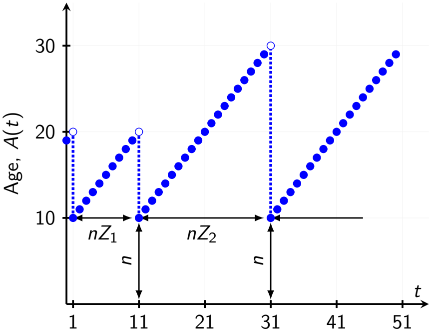
$$S(T_i) = \sum_{t=S_{i-1}}^{S_i - 1} A(t)$$

  is also *iid*

- By renewal reward theorem, the limiting average age is

$$\mathbb{E}A \triangleq \lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} A(s) = \mathbb{E}S(T_i)/\mathbb{E}T_i.$$

# Age Sample Path




- Inter-renewal time $T_i = nZ_i$
- Number of true update in $i$th renewal interval $Z_i$
- $\{Z_i : i \in \mathbb{N}\}$ is *iid* geometric with success parameter $(1 - p_1)$

# Mean Age

### Theorem

Limiting average age for the true update scheme is a.s.

$$\mathbb{E}A \triangleq \lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} A(s) = (n-1)/2 + n/(1-p_1).$$

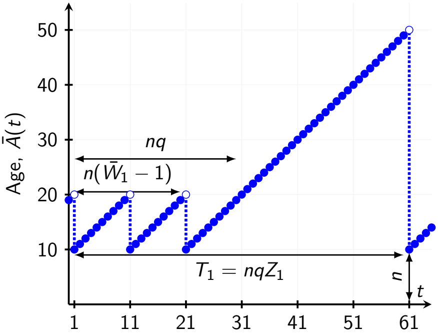
### Proof.

Cumulative age for $i$th renewal interval is

$$S(nZ_i) = \sum_{j=0}^{nZ_i-1} (n+j) = n^2 Z_i + nZ_i(nZ_i-1)/2.$$

$\square$

# Age Sample Path




- Inter-renewal time $T_i = nqZ_i$
- Number of successfully decoded contiguous incremental updates $\bar{W}_i - 1$ in the $i$th interval

# Incremental Upadate without Feedback

### Lemma

For each renewal interval, the number of successful receptions $1 \leq \bar{W}_i \leq q$, and is independent of the number of true updates $Z_i$. Further, the sequence $\{\bar{W}_i : i \in \mathbb{N}\}$ is *iid* and distributed as truncated geometric

$$\Pr\{\bar{W}_i = k\} = \begin{cases} (1-p_2)^{k-1}p_2, & 1 \leq k < q, \\ (1-p_2)^{q-1}, & k = q. \end{cases}$$

# Mean Age

## Theorem

Limiting average age for the incremental updates without feedback is

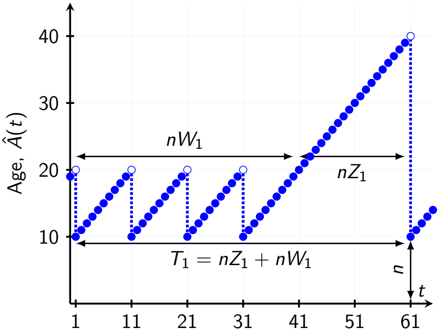
$$\mathbb{E}\bar{A} \triangleq \lim_{t\to\infty} \frac{1}{t}\sum_{s=1}^{t} \bar{A}(s) = \frac{\mathbb{E}\,T_i^2}{2\mathbb{E}\,T_i} + \frac{n^2\mathbb{E}\,\bar{W}_i(\bar{W}_i - 1)}{2\mathbb{E}\,T_i} - \left(n\mathbb{E}(\bar{W}_i - 2) + \frac{1}{2}\right).$$

## Proof.

Cumulative age $S(T_i)$ in the $i$th renewal interval is

$$S(T_i) = \sum_{j=1}^{\bar{W}_i - 1}\sum_{k=0}^{n-1}(n+k) + \sum_{j=n(\bar{W}_i-1)}^{T_i - 1}(n + j - n(\bar{W}_i - 1)),$$
$$= \frac{n^2\bar{W}_i(\bar{W}_i - 1)}{2} + \frac{T_i^2}{2} - \left(n(\bar{W}_i - 2) + \frac{1}{2}\right) T_i.$$

# Age Sample Path




- Inter-renewal time $T_i = nZ_i + nW_i$
- Number of incremental updates $W_i$ in $i$th renewal interval
- $\{W_i : i \in \mathbb{N}\}$ are *iid* geometric with success parameter $p_2$

# Mean Age

### Theorem

Limiting average age for the incremental updates with feedback is

$$\mathbb{E}\hat{A} \triangleq \lim_{t\to\infty} \frac{1}{t}\sum_{s=1}^{t} \hat{A}(s) = \frac{(3n-1)}{2} + \frac{n(\mathbb{E}Z_i^2 + \mathbb{E}Z_i)}{2(\mathbb{E}W_i + \mathbb{E}Z_i)}.$$

### Proof.

Cumulative age $S(T_i)$ over the $i$th renewal period $T_i$ is

$$\begin{aligned} S(T_i) &= \sum_{j=1}^{W_i-1}\sum_{k=0}^{n-1}(n+k) + \sum_{k=0}^{T_i-n(W_i-1)-1}(n+k) \\ &= (3n-1)T_i/2 + n^2(Z_i+1)Z_i/2. \end{aligned}$$

$\square$

# Analytical Comparison

**Theorem**

The mean age for the three schemes satisfy,

$$\mathbb{E}\hat{A} \leq \mathbb{E}A \leq \mathbb{E}\bar{A}.$$

# Numerical Comparision

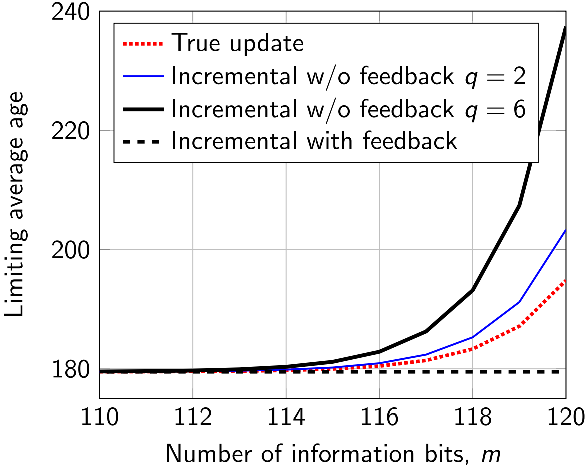
## Random Coding Scheme

- Probability of decoding failure is

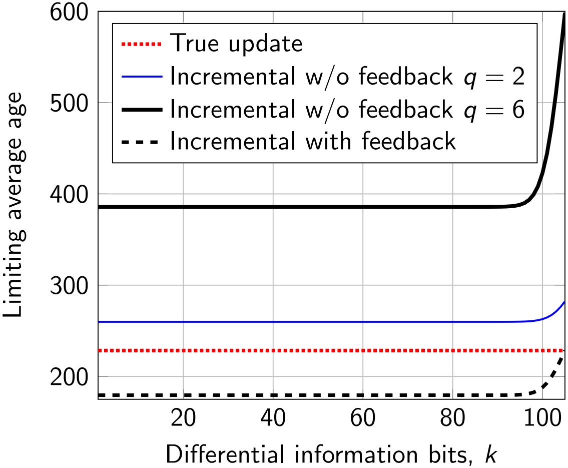
$$P(n, n-r, E) = 1 - \prod_{i=0}^{E-1} \left(1 - 2^{i-(n-r)}\right).$$

- Number of erasure $E$ has binomial distribution with parameter $(n, \epsilon)$
- For true update $p_1 = \mathbb{E}P(n, n-m, E)$ and incremental update $p_2 = \mathbb{E}P(n, n-k, E)$.

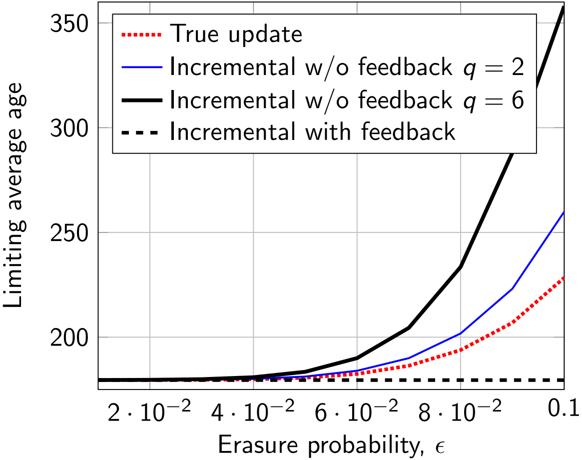
$n = 120$, $k = 10$, $\epsilon = 0.001$ and $q \in \{2, 6\}$



$n = 120,\ m = 105,\ \epsilon = 0.1$ and $q \in \{2, 6\}$



$n = 120$, $m = 105$ and $q \in \{2, 6\}$



# Discussion and Concluding Remarks

## Main Contributions

- Integration of coding and renewal techniques to study timely communication for delay-sensitive traffic
- We model channel unreliability by the erasure channel
- Incremental updates only when there is feedback availability

## Avenues of Future Research

- Extend results to structured sources
- Extend results to correlated finite-state erasure and error channels
- Impact of other coding schemes on timeliness