

The Power of Two in Large Service-Marketplaces

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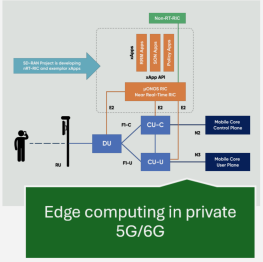
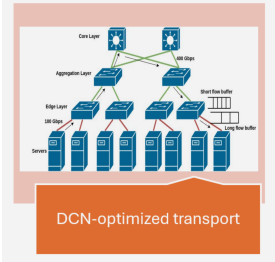
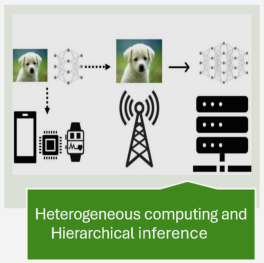
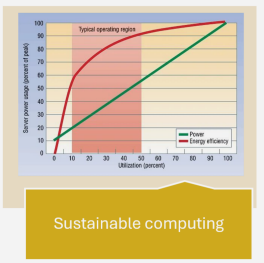
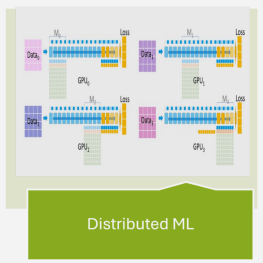
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भारतीय विज्ञान संस्थान



Distributed Systems



Acknowledgements



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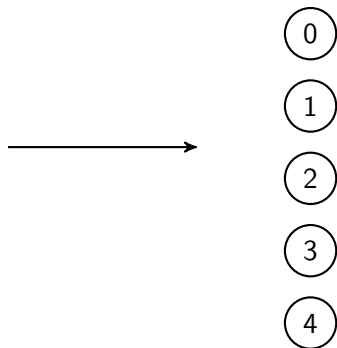
Qualcomm



National
Research
Foundation



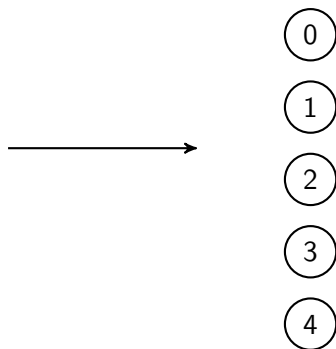
Problem setup



Questions

- ▶ **Routing:** How to route arriving tasks to minimize loss?
- ▶ **Pricing:** How to price the service to maximize revenue?

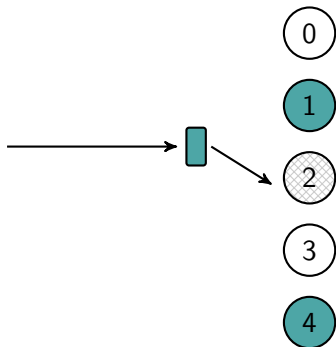
System model



N -server loss system

- ▶ Random *i.i.d.* unit mean exponential service times
- ▶ Poisson arrivals of rate $N\lambda$
- ▶ Server n is busy or idle denoted $X_n(t)$
- ▶ Random *i.i.d.* valuation with distribution G for each task

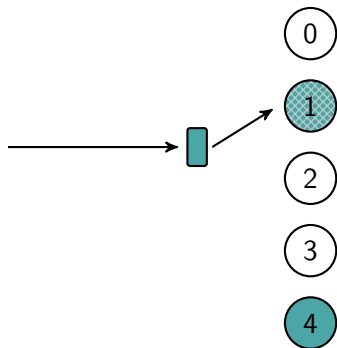
Deterministic routing D_1



Join an empty server

- ▶ Requires state information from all servers
- ▶ Loss only when all servers are busy
- ▶ Revenue if price less than valuation

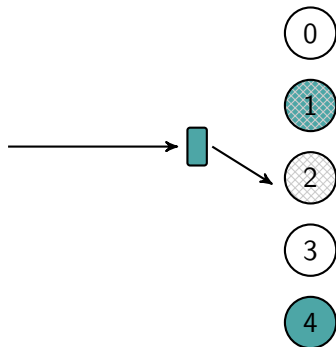
Random routing R_1



Join a random server

- ▶ Requires no server state feedback
- ▶ Loss when a busy server is selected
- ▶ No revenue can be generated

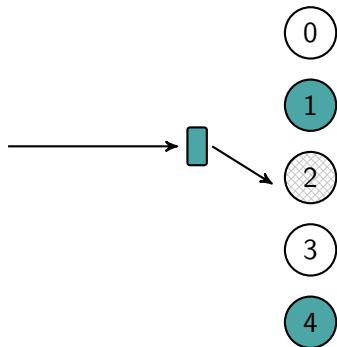
Power-of-two routing R_2



Join one of two randomly selected servers

- ▶ Requires server state feedback from two servers at each arrival
- ▶ Loss when both busy servers are selected
- ▶ No revenue if both servers are busy

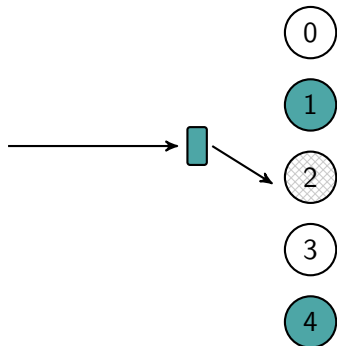
Pricing



Centralized and deterministic

- ▶ Centrally decided for all the servers
- ▶ Decided by individual servers
- ▶ Deterministic versus random

Centralized pricing for deterministic routing

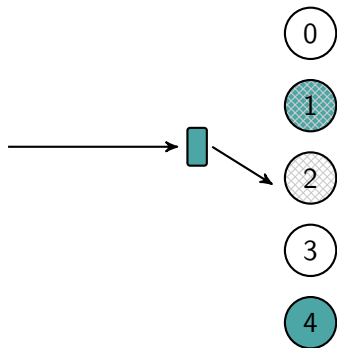


D₁C: State dependent pricing

- ▶ Revenue maximizing price given # busy servers ¹
- ▶ For large N state independent pricing maximizes revenue
- ▶ For price P at all servers, effective arrival rate $N\lambda\bar{G}(P)$
- ▶ For uniform pricing revenue rate per server is $\lambda P\bar{G}(P)$

¹ Ashok et al., "Optimal pricing in multi server systems," Performance Evaluation, vol. 154, p. 102282, 2022.

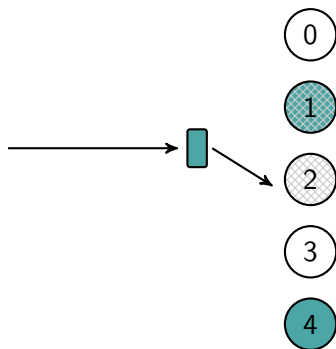
Decentralized pricing for power-of-2 routing



R_2G : Mean-field game

- ▶ Task joins the idle server with lower price if lower than value
- ▶ Each server picks its own price based on the empirical average of busy servers

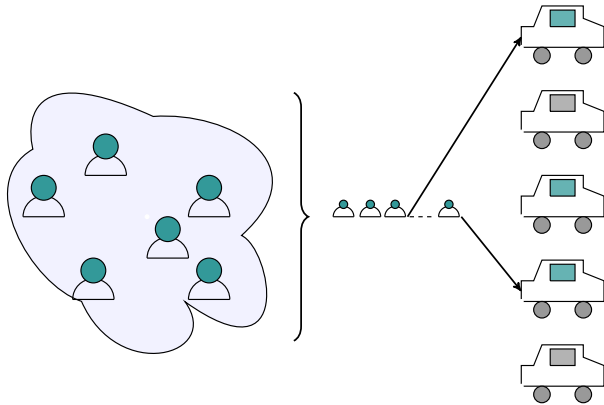
Problem Statement



R_2G : Mean-field game

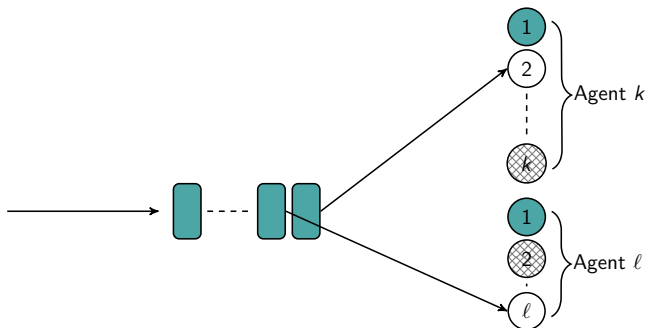
- ▶ Is there mean-field game equilibrium for this problem?
- ▶ Find the revenue rate under the mean-field game equilibrium

Ride sharing and on demand services



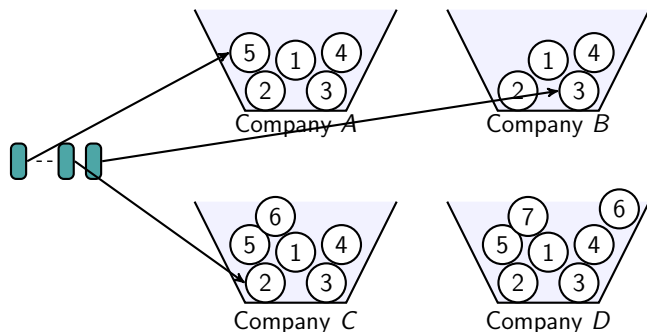
- ▶ Ride-hailing platforms like Uber and Lyft use dynamic pricing to match drivers with riders based on demand
- ▶ The two-server matching principle is similar to *two drivers competing for a ride* based on price and availability.

Online Cloud Marketplaces



- ▶ Google cloud and AWS marketplace allow independent cloud service providers to list their services
- ▶ Multiple providers compete for customer jobs, similar to the two-server price competition model

Online stock marketing



- ▶ Each conglomerate has a list of stocks whose prices vary
- ▶ We assume that these variations follow a specific distribution

State-of-the-art

Revenue maximizing dynamic pricing

- ▶ For a single server queue
 - ▶ Random valuation: [Naor, 1969]², [Borgs et al, 2011]³
 - ▶ Arbitrary valuation: [Ashok et al, 2023],⁴
- ▶ Multiple servers with no queues and random valuation
 - ▶ Centralized routing and pricing:[Ashok et al, 2022]⁵
 - ▶ **Our work: power-of-2 routing and rational pricing**

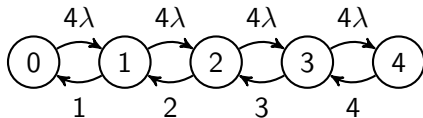
²P. Naor, "The regulation of queue size by levying tolls," *Econometrica*, vol. 37, no. 1, pp. 15–24, Jan. 1969.

³C. Borgs et al, "The optimal admission threshold in observable queues with state dependent pricing," *Probability in the Engineering and Informational Sciences*, vol. 28, no. 1, p. 101–119, 2014.

⁴Ashok et al., "Optimal pricing in a single server system," *ACM Trans. Model. Perform. Eval. Comput. Syst.*, vol. 8, no. 4, pp. 1–32, Dec. 2023.

⁵Ashok et al, "Optimal pricing in multi server systems," *Performance Evaluation*, vol. 154, p. 102282, 2022.

Deterministic routing D_1



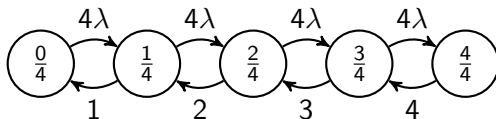
Number of busy servers $\sum_{n=1}^N X_n(t)$

- Evolve as a continuous time Markov chain with

$$Q_{x,x-1} = x,$$

$$Q_{x,x+1} = N\lambda$$

Deterministic routing D_1



Fraction of busy servers $Z(t) \triangleq \frac{1}{N} \sum_{n=1}^N X_n(t)$

- Evolve as a continuous time Markov chain with

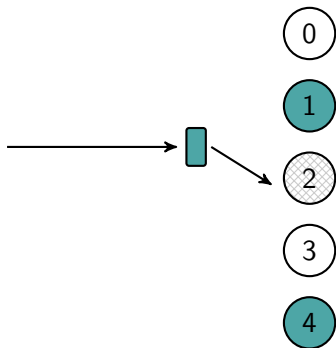
$$Q_{z, z - \frac{1}{N}} = Nz, \quad Q_{z, z + \frac{1}{N}} = N\lambda$$

- Mean rate of change of fraction of busy servers is

$$f(z) \triangleq \sum_w Q_{z,w}(w - z) = \lambda - z$$

- Mean-field limit $\frac{dz}{dt} \approx f(z) = \lambda - z$
 - If $\lambda < 1$, then stationary fraction $z^* = \lambda$
 - If $\lambda > 1$, then stationary fraction $z^* = 1$

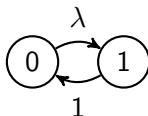
Centralized pricing for deterministic routing



D_1C and uniform pricing

- ▶ Effective arrival rate $\lambda \bar{G}(P)$ for common price P
- ▶ If $\lambda \bar{G}(P) < 1$, then revenue rate is $\lambda P \bar{G}(P)$
- ▶ If $\lambda \bar{G}(P) > 1$, then revenue rate is P maximized at $\bar{G}^{-1}(1/\lambda)$

Random routing R_1



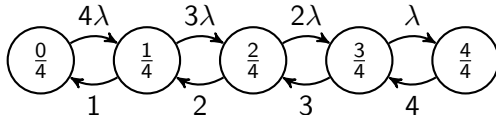
Number of busy servers

- Each server evolves independently as a continuous time Markov chain with

$$Q_{1,0} = 1,$$

$$Q_{0,1} = \lambda$$

Random routing R_1



Fraction of busy servers

- Evolve as a continuous time Markov chain with

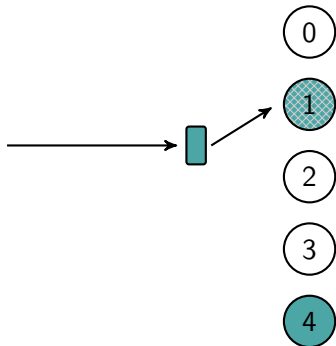
$$Q_{z, z - \frac{1}{N}} = Nz, \quad Q_{z, z + \frac{1}{N}} = N\lambda(1 - z)$$

- Mean rate of change of fraction of busy servers is

$$f(z) \triangleq \sum_w Q_{z,w}(w - z) = \lambda(1 - z) - z$$

- Mean-field limit $\frac{dz}{dt} \approx f(z) = \lambda(1 - z) - z$
 - Stationary fraction $z^* = \frac{\lambda}{1+\lambda}$

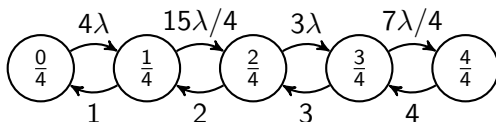
Centralized pricing for random routing



R_1C and uniform pricing

- ▶ Effective arrival rate $\lambda \bar{G}(P)$ for common price P
- ▶ Stationary fraction $z^* = \frac{\lambda \bar{G}(P)}{1 + \lambda \bar{G}(P)}$
- ▶ Revenue rate is $\lambda(1 - z^*)P \bar{G}(P)$

Power of two routing R_2



Fraction of busy servers

- Evolve as a continuous time Markov chain with

$$Q_{z, z - \frac{1}{N}} = Nz, \quad Q_{z, z + \frac{1}{N}} = N\lambda(1 - z^2)$$

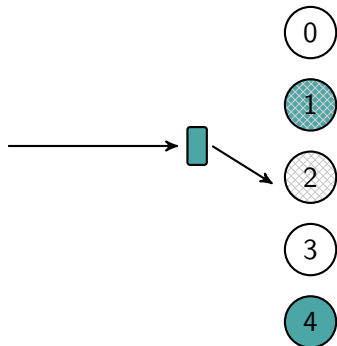
- Mean rate of change of fraction of busy servers is

$$f(z) \triangleq \sum_w Q_{z,w}(w - z) = \lambda(1 - z^2) - z$$

- Mean-field limit $\frac{dz}{dt} \approx f(z) = \lambda(1 - z^2) - z$

- Stationary fraction $z^* = -\frac{1}{2\lambda} + \sqrt{1 + \frac{1}{4\lambda^2}}$

Centralized pricing for power of two routing



R_2C and uniform pricing

- ▶ Effective arrival rate $\lambda \bar{G}(P)$ for common price P
- ▶ Stationary fraction $z^* = -\frac{1}{2\lambda \bar{G}(P)} + \sqrt{1 + \frac{1}{4\lambda^2 \bar{G}(P)^2}}$
- ▶ Revenue rate is $\lambda(1 - z^{*2})P \bar{G}(P)$

Mean-field game

Approach

- ▶ Valuation distribution is exponential with rate v
- ▶ Servers in $[N]$ follow same pricing, *i.i.d.* exponential price with rate d_1
- ▶ Find mean-field limit $z^*(d_1)$ of the fraction of busy servers $Z_t^N \triangleq \frac{1}{N} \sum_{n=1}^N X_{t,n}$, i.e. $z^*(d_1) = \lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} Z_t^N$
- ▶ Tag server 0 that has exponential price with rate d_0
- ▶ Find revenue rate of server 0 given d_1
- ▶ Choose best response rate $d_0^*(d_1)$ that maximizes revenue rate of server 0
- ▶ Is there a mean field game equilibrium?
- ▶ What is the per server revenue rate at this equilibrium?

System evolution

Admission indicators

- ▶ For k th arrival: task valuation V_k , price $P_{k,n}$ at server n
- ▶ Admission indicators

$$\begin{aligned}\eta_{k,10} &\triangleq \mathbb{1}_{\{V_k > P_{k,0}\}}, & \eta_{k,20} &\triangleq \mathbb{1}_{\{V_k > P_{k,0}, P_{k,0} < P_{k,n}\}}, \\ \zeta_{k,1} &\triangleq \mathbb{1}_{\{V_k > P_{k,n}\}}, & \zeta_{k,2} &\triangleq \mathbb{1}_{\{V_k > P_{k,n} \wedge P_{k,m}\}}.\end{aligned}$$

- ▶ Admission probabilities

$$q_1 \triangleq \mathbb{E}\eta_{k,10}, \quad q_{20} \triangleq \mathbb{E}\eta_{k,20}, \quad p_1 \triangleq \mathbb{E}\zeta_{k,1}, \quad p_2 \triangleq \mathbb{E}\zeta_{k,2}.$$

Evolution

Selection indicator for tagged server 0 by the k th task

$$\xi_k^N = \mathbb{1}_{\{0 \in I_k\}} \bar{X}_{A_k,0} \sum_{n=1}^N \mathbb{1}_{\{n \in I_k\}} \left(X_{A_k,n} \eta_{k,10} + \bar{X}_{A_k,n} \eta_{k,20} \right).$$

System evolution

Generator matrix

The process $(X_{t,0}, Z_t^N)$ is a CTMC with the generator matrix Q^N defined as

$$Q_{(x,z),(y,w)}^N = \begin{cases} Nz, & w = z - \frac{1}{N}, y = x \\ \lambda \bar{z}(2p_1(x + Nz) + 2\bar{x}q_{21} + p_2(N\bar{z} - 1)), & w = z + \frac{1}{N}, y = x \\ x, & w = z, y = x - 1, \\ 2\lambda \bar{x}(zq_1 + \bar{z}q_{20}), & w = z, y = x + 1. \end{cases}$$

McKean-Vlasov equation

Consider an autonomous dynamic system $\dot{z} = h(z)$, where

$$h(z) \triangleq \lim_{N \rightarrow \infty} \sum_{y,w} Q_{(x,z),(y,w)}^N (w - z) = \lambda \bar{z}(2zp_1 + \bar{z}p_2) - z.$$

Limiting fraction of busy servers

Let $\alpha \triangleq \frac{\nu}{d_1}$ and $x \triangleq \frac{1}{2} \left(\alpha + \frac{(1+\alpha)(2+\alpha)}{2\lambda} \right)$, then the unique rest point z^* such that $h(z^*) = 0$ is

$$z^* \triangleq -x + \sqrt{1 + \alpha + x^2}.$$

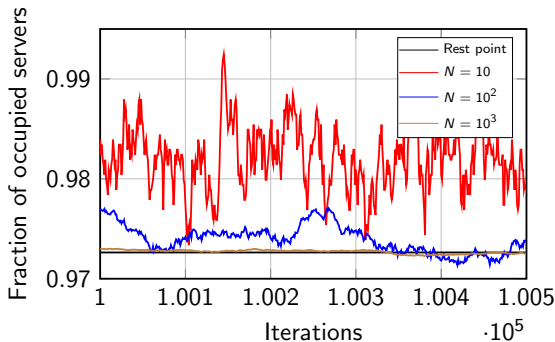
Our Contributions

- ▶ Calculated the deterministic occupancy z^* of the sub-system using McKean-Vlasov equation
- ▶ Derived the tagged server's limiting revenue expression as a function of z^* , price and value rates
- ▶ Designed an algorithm that plays a game between the agents to choose the optimum price parameter which maximizes their revenue
- ▶ Derived the numerical results for mean price, limiting revenue and throughput of ours' as well as the state-of-art techniques and compared them

Mean-field convergence

Mean-field convergence

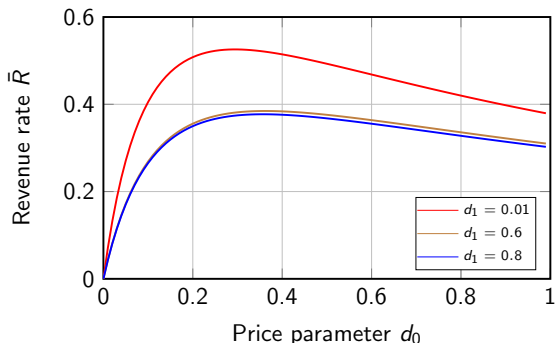
The stationary fraction of busy servers Z_∞^N converges in the mean-square sense to unique rest point z^* of mean-field model with rate $\frac{1}{N}$. That is, $\lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{E} |Z_t^N - z^*|^2 = O\left(\frac{1}{N}\right)$



Tagged server revenue

Limiting revenue rate at tagged server 0

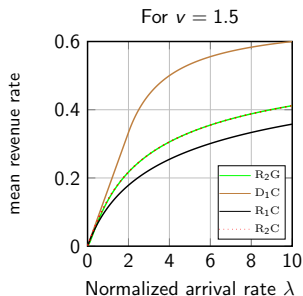
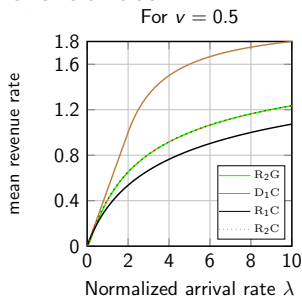
$$\bar{R} \triangleq \lim_{N \rightarrow \infty} \lim_{K \rightarrow \infty} \frac{1}{A_K} \sum_{k=0}^K P_{k,0} \xi_k^N = \frac{z^* q_1^2 + (1-z^*) q_{20}^2}{d_0 (\frac{1}{2\lambda} + z^* q_1 + (1-z^*) q_{20})}$$



- ▶ We can show $d_0 \mapsto z^* \mapsto d_0^*$ is composition of continuous maps
- ▶ There exists a fixed-point which is the mean-field game equilibrium

Performance comparison

Mean revenue rate



- ▶ D_1C has the best revenue rate at the cost of highest server feedback
- ▶ R_2G has same performance as R_2C without coordinated pricing
- ▶ R_1C has the worst performance since it is completely agnostic of system state

Conclusion

- ▶ Proved that the mean occupancy rate Z_t^N converges to the stationary point z^* and the simulated revenue rate converges to the calculated one
- ▶ Revenue rate is uni-modal over the tag price rate d_0 and proposed an algorithm which strategically chooses the price rates which maximizes the tagged server revenue rate
- ▶ Calculated the revenue rates, blocking probability and the mean tag price rate for the state-of-art techniques numerically and compared the results

References

- ▶ P. Naor, "The regulation of queue size by levying tolls," *Econometrica*, vol. 37, no. 1, pp. 15–24, Jan. 1969.
- ▶ C. Borgs et al, "The optimal admission threshold in observable queues with state dependent pricing," *Probability in the Engineering and Informational Sciences*, vol. 28, no. 1, p. 101–119, 2014.
- ▶ A. Krishnan K.S., C. K. Singh, S. T. Maguluri, and P. Parag, "Optimal pricing in multi server systems," *Performance Evaluation*, vol. 154, p. 102282, 2022.
- ▶ L. Ying, "On the approximation error of mean-field models," in *ACM SIGMETRICS Inter. Conf. Meas. Model. Comp. Sci.*, Jun. 2016, pp. 285–297.
- ▶ A. Krishnan K.S., C. K. Singh, S. T. Maguluri, and P. Parag, "Optimal pricing in a single server system," *ACM Trans. Model. Perform. Eval. Comput. Syst.*, vol. 8, no. 4, pp. 1–32, Dec. 2023.
- ▶ N. D. Vvedenskaya, R. L. Dobrushin, and F. I. Karpelevich, "Queueing system with selection of the shortest of two queues: An asymptotic approach," *Prob. Info. Transmission*, vol. 32, no. 1, pp. 15–27, 1996.
- ▶ D. Narasimha, S. Nomula, S. Shakkottai, and P. Parag, "The Power of Two in Large Service-Marketplaces", in *IEEE Conf. Comp. Commun. (INFOCOM)*, 2025, pp. 1–10.