E2 205 Error-Control Coding
Homework 1

Discussion: 2:00 PM, Aug 23, 2019

1. Let $\underline{x}, \underline{y} \in \mathbb{F}_2^{16}$ with $d_H(\underline{x}, \underline{y}) = 6$. What is the size of the intersection

$$B(\underline{x}, 4) \bigcap B(\underline{y}, 4)$$

where $B(\underline{z}, r)$ for $\underline{z} \in \mathbb{F}_2^{16}$, denotes the set of all vectors in $\mathbb{F}_2^{16}$ whose Hamming distance from $\underline{z}$ is $\leq r$?

2. Define the *intersection* of two binary vectors $\underline{x}$ and $\underline{y}$ to be the vector

$$\underline{x} * \underline{y} = (x_1 y_1, \, x_2 y_2, \, \cdots, \, x_n y_n)^t,$$

which has 1's only where both $\underline{x}$ and $\underline{y}$ do. For example

$$[1\ 1\ 0\ 0\ 1]^t * [1\ 0\ 1\ 1\ 1]^t = [1\ 0\ 0\ 0\ 1]^t.$$

Show that

$$w(\underline{x} + \underline{y}) = w(\underline{x}) + w(\underline{y}) - 2 \cdot w(\underline{x} * \underline{y})$$

where $w(\underline{z})$ denotes the Hamming weight of $\underline{z}$.

3. What is the smallest possible minimum distance of a block code of length n that can correct 2 errors and detect 5 errors ? If used only for error-detection, what is the maximum number of errors that the code can detect ?

4. A *ternary* code $\mathscr{C}$ is a code whose symbol alphabet is the set $\{0,1,2\}$, i.e., $\mathscr{C}$ is a subset of $\{0,1,2\}^n$. Even in $\{0,1,2\}^n$, the definitions of Hamming weight and Hamming distance remain as in the binary case. In the binary case, a code is a (t_d, t_c) code iff

$$d_{\min} \geq t_d + t_c + 1.$$

Is this also true in the ternary case ? (The definition of a (t_d, t_c) code remains as in the binary case.) Justify your answer.

5. Let A be an $(m \times n)$ matrix over the field $\mathbb{F}_2$. Given $\underline{a}, \underline{b} \in \mathbb{F}_2^n$, we define $\underline{a} \sim \underline{b}$ if

$$A(\underline{a} - \underline{b}) = \underline{0}.$$

Prove that this is an equivalence relation. How many equivalence classes are there if A has rank r ?

6. Define two binary vectors of length $2m$, $\underline{z}_1 = (\underline{x}_1, \underline{y}_1)$, and $\underline{z}_2 = (\underline{x}_2, \underline{y}_2)$ to be equivalent iff

$$\begin{aligned} \underline{x}_1 &= \underline{x}_2 \quad \text{and} \\ d_H(\underline{y}_1, \underline{y}_2) &\quad \text{is} \quad \text{even}. \end{aligned}$$

(a) Show that this is an equivalence relation on the set $\mathbb{F}^{2m}$.

(b) Into how many equivalence classes does this partition $\mathbb{F}^{2m}$?

7. Provide an example of a non-Abelian group.

8. Let $G = \mathbb{F}_2^5$. Let H be the subset of G consisting of the four 5-tuples shown below:

$$H = \left\{ \begin{array}{c} (\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )^t \\ (\ 0\ \ 1\ \ 1\ \ 1\ \ 0\ )^t \\ (\ 1\ \ 0\ \ 1\ \ 0\ \ 1\ )^t \\ (\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ )^t \end{array} \right\}$$

Verify that H is a subgroup of G and identify all the cosets of H in G.

9. Let Z_{12} denote the set of integers modulo 12 under modulo 12 addition. Identify all subgroups of $G = Z_{12}$.