E2 205 Error-Control Coding
Homework 5

1. In the computation:

$$\beta(x_3) = \sum_{x_1,\, x_2,\, x_4,\, x_5} f(x_1)g(x_2)h(x_1,x_2,x_3)p(x_3,x_4)q(x_3,x_5),$$

   all the variables $x_i, i = 1,2,3,4,5$ take on values from an alphabet $\mathscr{A}$ of size $|\mathscr{A}| = q$. If you were to reorganize this expression to minimize the number of operations (additions and multiplications), how would you do it and how many operations would you end up needing?

2. Consider the "min-star" semi-ring $((-\infty,\infty],\ \min^*,\ +)$ in which the $\min^*$ operation is given by:

$$\overset{*}{\min}(x,y) := -ln(e^{-x}+e^{-y}).$$

   (a) Identify the identity element under the min-star operation.

   (b) Verify that the distributive law holds.

3. Consider the problem of computing

$$F(x_1) = \sum_{x_2=0}^{9}\sum_{x_3=0}^{9}\sum_{x_4=0}^{9} f(x_2,x_3,x_4)g(x_3,x_4)h(x_1,x_2,x_4).$$

   The functions $f(\cdot)$, $g(\cdot)$,$h(\cdot)$, are all real-valued functions.

   (a) It is desired to pose this problem as a marginalize a product function problem. Identify the corresponding universal set, the corresponding local domains and the local and global kernels.

   (b) Organize if possible these local domains into a junction tree. Make clear all your working.

4. Determine the savings in computation between using the brute force approach to computing

$$G(x,y) = \sum_{u\in\mathbb{F}_q}\sum_{v\in\mathbb{F}_q}\sum_{w\in\mathbb{F}_q}\sum_{z\in\mathbb{F}_q} f(u,w,x,z)g(v,y,z)h(u,x,z)p(x,y,z)$$

   versus the approach that makes intelligent use of the distributive law. The functions f, g, h, p are all real-valued functions.

5. Verify that the *min** sum semi-ring is in fact a semi-ring, starting from the defintion of the *min** operation:

$$min^*(x,y) = min\{x,y\} - ln(1+e^{-|y-x|}) = -ln(e^{-x}+e^{-y}).$$

   Identify the underlying set and the identity element under the *min** operation.

6. Consider the single parity-check code of length 3 having parity check matrix $H = [1\ 1\ 1]$. Thus $\underline{v}^T = [v_1,\ v_2,\ v_3]$ is a codeword if and only if $H\underline{v} = 0$. In a certain instance, when communicating over a binary symmetric channel (BSC) having crossover probability $\varepsilon < 0.5$, the received vector was found to be

$$\underline{y}^T = [0\ 1\ 0].$$

Use the GDL to carry out ML code-symbol decoding of this code. Show all intermediate steps

(a) the formulation as an MPF problem,

(b) the junction tree,

(c) the message-passing schedule and the messages passed,

(d) the result of decoding.

7. The junction tree of the $[7,4,2]$ code with parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

is shown in Fig. 1.

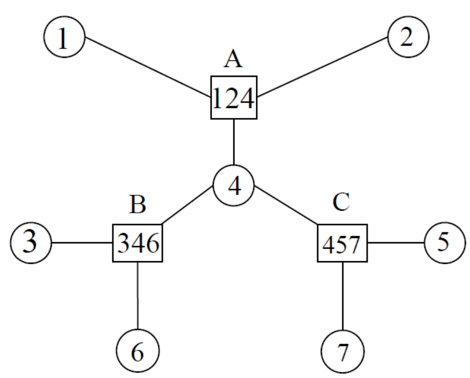

Figure 1: Junction tree of the $[7,4,2]$ code.

Assume that the GDL is used to compute the objective function at node 7 and orient all edges in the tree towards node 7. Interpret the message being passed from node 4 to node C as a belief. Identify this belief as well as the underlying evidence that it is based upon, as *precisely* as you can. Explain your answer. 4 points

8. Consider decoding the $[7,4,2]$ linear block code for the case when the received vector across a binary symmetric channel with crossover probability $\varepsilon << 1$, is the vector

$$\underline{y} = [1\,0\,0\,0\,0\,0\,0]^T.$$

Use the GDL to make decisions based on maximizing the aposteriori probabilities

$$p(u_i|\underline{y})$$

of the code symbols $u_i, i = 1,2,3,4,5,6,7$.

9. Is it possible to decode the $[7,4,3]$ binary Hamming code having parity-check matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

using the standard GDL solution to the associated MPF problem ? Explain clearly showing all your working.

10. Consider the rate $1/2$ convolutional code with

$$G(D) = [1+D+D^2 \quad 1+D^2].$$

If the received sequence (across a discrete memoryless AWGN channel) is

$$\underline{r} = (4 \; -1 \quad -3 \; 2 \quad 6 \; -5),$$

use the GDL algorithm to implement minimum probability of bit error decoding of the message bits u_0, u_1, u_2. Show all your working including the graphs that you use and the message passing schedule.

11. Consider the joint probability function

$$p(\{u_i\}_{i=0}^3, \{s_i\}_{i=0}^4, \{y_i\}_{i=0}^3) = p(s_0)\prod_{i=0}^{3} p(u_i)p(s_{i+1}|s_i,u_i)p(y_i|s_i,u_i)$$

associated with a convolutional code. Here $\{u_i \in \{0,1\}\}$ represent the binary message symbols, the $\{s_i\}$ is the state sequence and $\{y_i\}$ are the received symbols. Consider the problem of maximum-likelihood code-symbol decoding of this code, i.e., of computing $p(u_k/\{y_i\}_{i=0}^3)$, $0 \le k \le 3$.

(a) Present this as an MPF problem,

(b) organize the local domains into a junction tree

(c) show that message passing can be organized into a forward wave and a backward wave and that the forward wave is in essence, a sequence of matrix multiplications