Error Correcting codes **Date : 30-09-2019**

Lecture 14: Finite Block Length Bounds

Lecturer: P. Vijay Kumar *Scribed by: Ashwitha*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

14.1 Recap : Noisy Channel Coding Theorem

We have shown that

$$P_{e,m} \leq exp\left(-N \max_{\rho,Q}\left(E_o\left(\rho,Q\right)-\rho R\right)\right)$$

Typically one chooses Q to result in $I\left(Q,P\right)=$ Capacity of the DMC. Then, next given this value of Q, one chooses ρ to then minimize the RHS of the above inequality. This minimization has a graphical interpretation as discussed in the previous lecture.

Turns out $\frac{\partial^2 E_o(\rho,Q)}{\partial \rho^2} \leq 0$

$Eo\left(\rho,Q\right)\geq 0$

$E_0\left(\rho,Q\right)=0$, $\rho=0$, $\frac{\partial E_o(\rho,Q)}{\partial \rho}|_{\rho=0}=I\left(\rho,Q\right)$

$E_0\left(\rho,Q\right)$ is a concave function for $\rho \geq 0$

$$E_r\left(R\right)=\max_{0\leq\rho\leq 1}\max_{Q}\left[E_o\left(\rho,Q\right)-\rho R\right]$$

is the random coding exponent.

$$P_{e,m}\leq exp\left(-NE_r\left(R\right)\right), 1\leq m\leq M$$

$$P_e \leq exp\left(-NE_r\left(R\right)\right)$$

Since $E_r\left(R\right)>0$ for $R<C$, this proves that using very large block lengths, reliable communication is possible at all rates R, $0\leq R<C$.

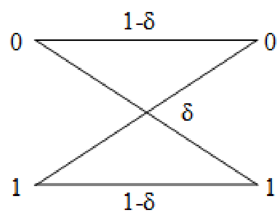
With $Q\left(.\right)$ selected to make $I\left(Q,P\right)=C$,

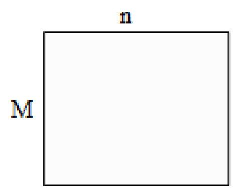
$$R_{cr}\triangleq\frac{\partial E_o\left(\rho,Q\right)}{\partial\rho}|_{\rho=1}$$

the optimizing value of $\rho=1$

R_{cr} is called critical rate and turns up in other contexts as well. It is also called the computational cut-off rate.

14.2 Finite blocklength bound for BSC



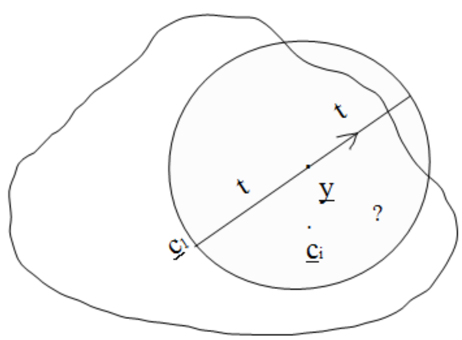


14.2.1 Random coding union bound

All symbols in the code book are $Bern\left(\frac{1}{2}\right)$ iid.

$$\underline{y} = \underline{c}_1 + \underline{z}$$

$Pr\left(error \mid \underline{y}, \underline{c}_1, w_H\left(\underline{z}\right) = t\right) \leq$ probability that some other code word lies in $B\left(\underline{y}, t\right)$



$$Pr\left(error \mid \underline{y}, \underline{c_1}, w_H\left(z\right) = t\right) \leq min\left\{1, \sum_{i=2}^{M} \frac{\sum_{i=0}^{t}\binom{n}{i}}{2^n}\right\} \leq min\left\{1, 2^{nR}\sum_{i=0}^{t}\binom{n}{i}2^{-n}\right\}$$

$$Pr\left(error \mid \underline{y}, \underline{c}_1 \; transmitted\right) = \left(\sum_{t=0}^{n}\binom{n}{t}\delta^t\left(1-\delta\right)^{n-t}\right) min\left\{1, \sum_{i=0}^{t}\binom{n}{i}2^{-n(1-R)}\right\}$$

Set for $j \geq 2$

$$A_j = Pr\left(\underline{c}_j \in B\left(\underline{y}, t\right)\right)$$

$$P\left(\bigcup_{j=2}^{M} A_j\right) \leq \left(\sum_{j=2}^{M} P\left(A_j\right)\right)^{\rho} \quad \text{(Gallager)}$$

Here

$$P\left(\bigcup_{j=2}^{M} A_j\right) \leq min\left\{1, \sum_{j=2}^{M} P_r\left(A_j\right)\right\}$$

(Yury Polyanskiy, H. Vincent Poor, Sergio Verdu, "Channel Coding Rate in the Finite Blocklength Regime", IEEE Transactions on Information Theory, April 2010)

The RCU bound is an upper bound to the average probability of codeword error, averaged over the ensemble of codes. Hence there must be some code in this ensemble having codeword error probability that is less than or equal to this average.

Now, we will derive a lower bound to the error probability of any code.

Let $\mathcal{C}$ be a block code of length n containing M code words.

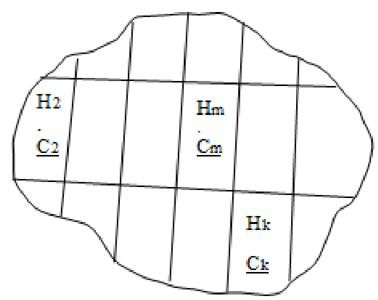


Figure 14.1: Space of all received vectors

Let $A_{\ell,m}$ denote the number of received vectors at hamming distance ℓ from codeword m that are decoded to codeword $\underset{\sim}{c}_m$, $1 \leq m \leq M$; $0 \leq \ell \leq n$.

$$1 - P_e = Pr\left(\texttt{correct decision}\right))$$

$$= \frac{1}{M}\sum_{m=1}^{M}\left(\sum_{\ell=0}^{n} A_{\ell,m}\delta^{\ell}(1-\delta)^{n-\ell}\right)$$

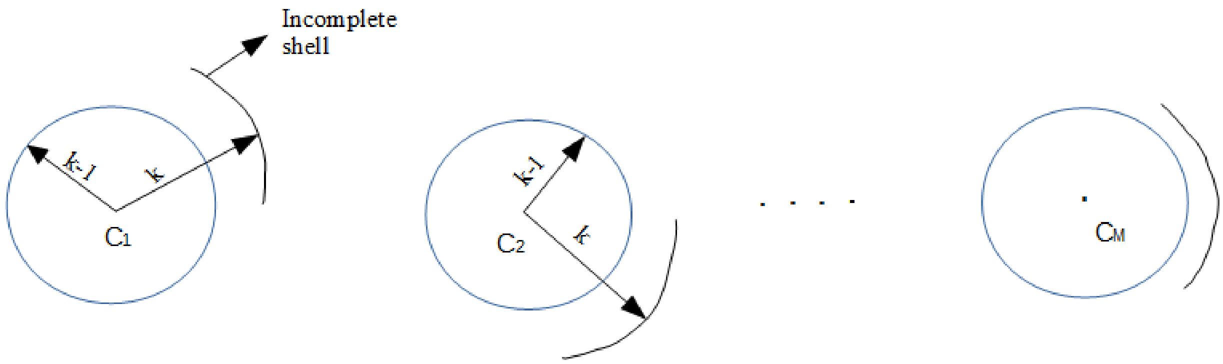
$$P_e = \frac{1}{M}\sum_{m=1}^{M}\sum_{\ell=0}^{n}\left(\binom{n}{\ell} - A_{\ell,m}\right)\delta^{\ell}(1-\delta)^{n-\ell}$$

Consider the probability of choosing $A_{\ell,m}$ to minimize the RHS of above equation.

Let k be such that

$$M\sum_{i=0}^{k-1}\binom{n}{i} + \sum_{m=1}^{M} A_{k,m} = 2^n$$

$k-1 =$ largest radius of the ball such that M(size of the ball) $\leq 2^n$



Claim: P_e is minimized by setting

$$A_{\ell,m} = \begin{cases} \binom{n}{\ell}, & 0 \leq \ell \leq k-1 \\ 0, & k+1 \leq \ell \leq n \end{cases} \tag{14.1}$$

and choosing $A_{k,m}$ such that

$$M \sum_{\ell=0}^{k-1} A_{\ell,m} + \sum_{m=1}^{M} A_{k,m} = 2^n$$

This is apparent by examining the above figure.

Also

$$\sum_{m=1}^{M} \sum_{k=0}^{n} A_{k,m} = 2^n$$

$$\begin{aligned}
P_e &\geq \frac{1}{M} \sum_{m=1}^{M} \sum_{\ell=k+1}^{n} \binom{n}{\ell} \delta^{\ell} (1-\delta)^{n-\ell} + \frac{1}{M} \sum_{m=1}^{M} \left(\binom{n}{k} - A_{k,m} \right) \delta^k (1-\delta)^{n-k} \\
&= \sum_{\ell=k+1}^{n} \binom{n}{\ell} \delta^{\ell} (1-\delta)^{n-\ell} + \left(1 - \frac{\sum_{m=1}^{M} A_{k,m}}{M \binom{n}{k}} \right) \binom{n}{k} \delta^k (1-\delta)^{n-k} \\
&= \sum_{l=k+1}^{n} \binom{n}{\ell} \delta^{\ell} (1-\delta)^{n-\ell} + \lambda \binom{n}{k} \delta^k (1-\delta)^{n-k}, \text{ where } \lambda = 1 - \frac{\sum_{m=1}^{M} A_{k,m}}{M \binom{n}{k}} \\
&= (1-\lambda) \sum_{\ell=k+1}^{n} \binom{n}{\ell} \delta^{\ell} (1-\delta)^{n-\ell} + \lambda \sum_{l=k}^{n} \binom{n}{\ell} \delta^{\ell} (1-\delta)^{n-\ell}
\end{aligned}$$