E2 205 Error-Control Coding
Lecture 15

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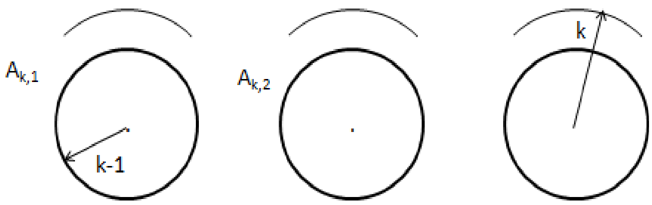
1 Sphere - Packing Bound

Given (n,M) let (k-1) be the largest integer such that

$$\boxed{M\sum_{i=0}^{k-1}\binom{n}{i} + \sum_{m=1}^{M} A_{k,m} = 2^n}$$



$$(1-p_e) \leq \frac{1}{M}\sum_{m=1}^{M}\sum_{i=0}^{k-1}\binom{n}{i}\delta^i(1-\delta)^{n-i} + \frac{1}{M}\sum_{m=1}^{M} A_{k,m}\binom{n}{k}\delta^k(1-\delta)^{n-k}$$

$$p_e \geq \frac{1}{M}\sum_{m=1}^{M}\sum_{i=k+1}^{n}\binom{n}{i}\delta^i(1-\delta)^{n-i} + \frac{1}{M}\sum_{m=1}^{M}(\binom{n}{k} - A_{k,m})\delta^k(1-\delta)^{n-k}$$

$$= E_k + \left[\binom{n}{k} - \frac{1}{M}(2^n - M\sum_{i=0}^{k-1}\binom{n}{i})\right]\delta^k(1-\delta)^{n-k}$$

where $E_k = \sum_{i=k+1}^{n}\binom{n}{i}\delta^i(1-\delta)^{n-i}$.

Note:

$$M\sum_{i=0}^{k-1}\binom{n}{i} > 2^n \implies 2^n - M\sum_{i=0}^{k-1}\binom{n}{i} < \binom{n}{k}M$$

$$p_e \geq E_k + (1-\lambda)\left[\binom{n}{k}\delta^k(1-\delta)^{n-k}\right], \text{ where } \lambda = \frac{2^n - M\sum_{i=0}^{k-1}\binom{n}{i}}{\binom{n}{k}M}$$

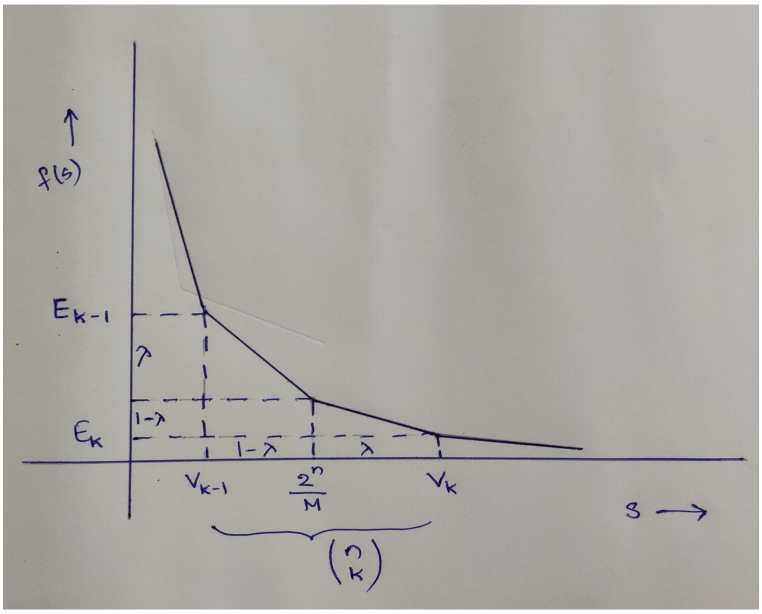
$= E_k + (1-\lambda)(E_{k-1} - E_k)$

$= \lambda E_k + (1-\lambda)E_{k-1}$

<u>Define:</u>

$f(\sum_{i=0}^{\ell} \binom{n}{i}) = \sum_{i=\ell+1}^{n} \binom{n}{i}\delta^i(1-\delta)^{n-i}$

LHS represents Volume of ball of radius ℓ (V_ℓ) and RHS represents probability that noise vector lies outside the ball (E_ℓ) in BSC.



$V_{k-1} + \lambda\binom{n}{k} = \sum_{i=0}^{k-1} \binom{n}{i} + \binom{n}{k}[\frac{2^n - M\sum_{i=0}^{k-1}\binom{n}{i}}{\binom{n}{k}M}]$

$= \sum_{i=0}^{k-1} \binom{n}{i} + [\frac{2^n - M\sum_{i=0}^{k-1}\binom{n}{i}}{M}]$

$= \frac{2^n}{M}$

$\therefore\ p_e \geq f(\frac{2^n}{M})$ (The Sphere-Packing Bound).

2 Convolutional Codes

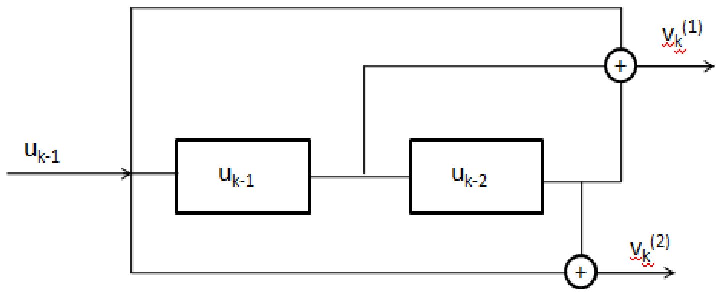


Figure 1: Convolutional Encoder

$v_k{}^{(1)} = u_k + u_{k-1} + u_{k-2}, k \geq 0$

$v_k{}^{(2)} = u_k + u_{k-2}, k \geq 0$

Assume $u_{-1} = u_{-2} = 0$ (shift register initialized to 00)

Convolutional codes are linear,time-invariant tree codes. Unlike the block codes, convolutional codes have finite memory.

An example of a tree code

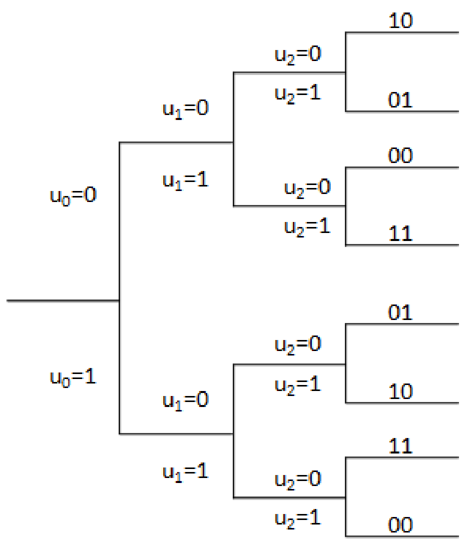


Figure 2: Tree code

u_k	u_{k-1}	u_{k-2}	$v_k^{(1)}$	$v_k^{(2)}$
0	0	0	0	0
1	0	0	1	1
0	0	1	1	1
1	0	1	0	0
0	1	0	1	0
1	1	0	0	1
0	1	1	0	1
1	1	1	1	0

2.1 Semi-infinite Generator matrix

$$G = \begin{bmatrix} 11 & 10 & 11 & & & \\ & 11 & 10 & 11 & & \\ & & 11 & 10 & 11 & \\ & & & 11 & 10 & .. \\ & & & & 11 & .. \\ & & & & & .. \end{bmatrix}$$

3 Formal Power Series

Let F be a field, then

$$F[[X]] = \{\sum_{k=0}^{\infty} a_k x^k | a_k \in F\}$$

called the Power Series (ring of formal power series)

$\sum_k a_k x^k + \sum_k b_k x^k = \sum_k (a_k + b_k) x^k$

$c \sum_k a_k x^k = \sum_k c a_k x^k$

$(\sum_{k=0}^{\infty} a_k x^k)(\sum_{l=0}^{\infty} b_l x^l) = \sum_{n=0}^{\infty} c_n x^n$

$c_n = \sum_{k+l=n} a_k b_l$ (Convolution)

$$\boxed{c_n = \sum_{k=0}^{n} a_k b_{n-k}}$$

4 Field of Laurent series

$F(x) = \{\sum_{k=-d}^{d} a_k x^k |$ d an integer, $a_k \in F\}$

Example:

- $(X^3 + X^7 + X^8)^{-1}$

  $= [X^3(1 + X^4 + X^5)]^{-1}$

$$= \frac{X^{-3}}{1+(X^4+X^5)}$$

$$= X^{-3}[1 - (X^4 + X^5) + (X^4 + X^5)^2 - ....]$$