E2 205 Error-Control Coding
Lecture 17

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1 Performance of Convolutional Codes

In the trellis diagram, the metric of a path is the sum of path metrics in the case of a BSC.
This also holds in the case of the AWGN.

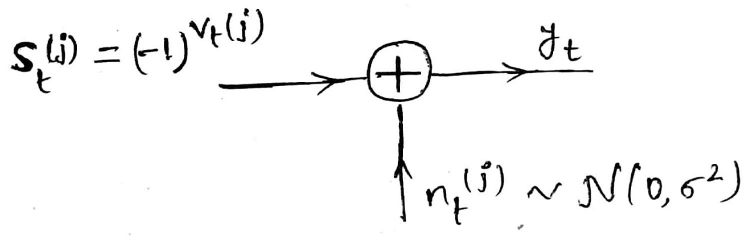


Fig. AWGN Channel

P$(\vec{y}|\vec{c_i}) = \prod_{t=0}^{N-1}\prod_{j=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\frac{-1}{2\sigma^2} \ (y_t(j) - s_t(j))^2)$.

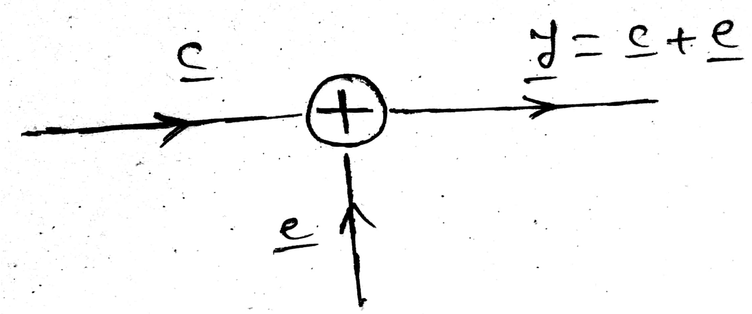
So, the MLD reduces to a minimum Eulidean distance decoder.

$\min \sum_{t=0}^{N-1} \sum_{j=1}^{n}(y_t(j) - s_t(j))^2$

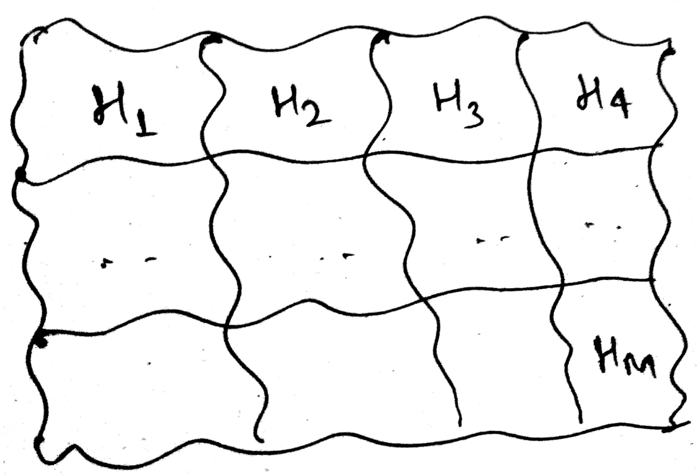
Thus it suffices to maximize
$\sum_{t=0}^{N-1} \sum_{j=1}^{2} y_t(j)s_t(j)$ (the inner product)

The rate of the procedure remains as before with min. replaced by max. Termination is not required in practice because the survivors at time $t+\tau$, tend to have a common prefix, up until time t, allowing code symbols upto time to be decoded by time $t+\tau$.

Consider the BSC in the form:



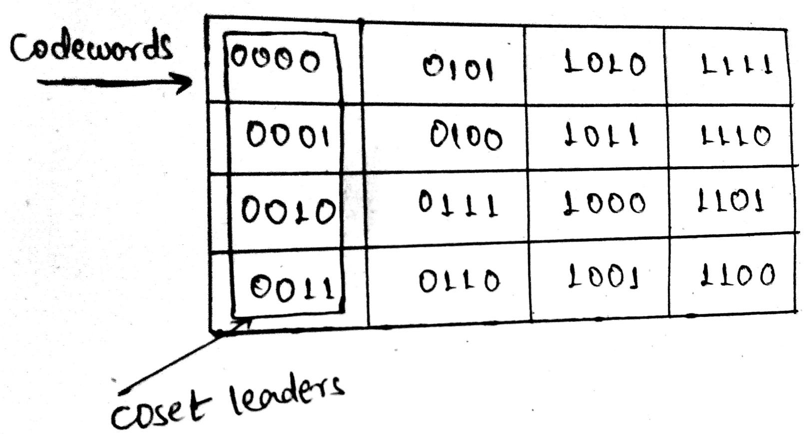
where $\vec{c} \in \mathcal{C}$ an [n,k] linear code.
At the receiver let the decision regions be $H_i, 1 \leq i \leq M = 2^k$.



$\vec{y} \in$ H$_i$ requires that $d_H(\vec{y},\vec{c_i}) \leq d_H(\vec{y},\vec{c_j})$.

Let us choose H_i s.t.
$\vec{y} \in$ H$_i$, iff $d_H(\vec{y},\vec{c_i}) \leq d_H(\vec{y},\vec{c_j})$.

Standard-Array Table



0000	0101	1010	1111
0001	0100	1011	1110
0010	0111	1000	1101
0011	0110	1001	1100

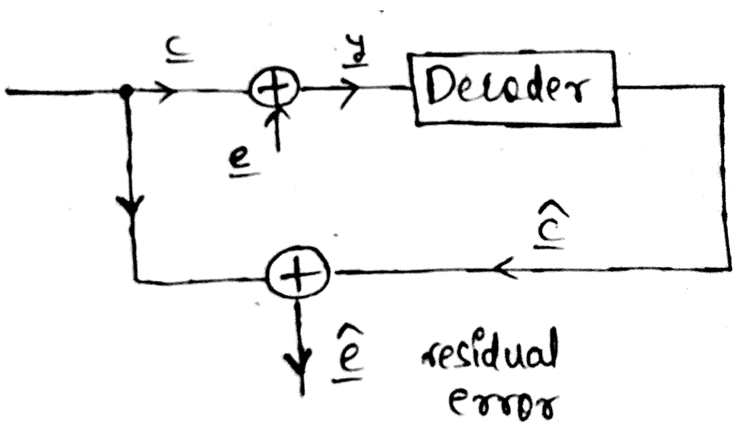
Let $E = ([0000]^t, [0001]^t, [0010]^t, [0011]^t)$ be the set of coset leaders.

$H_0 = E(i^{th}\ column\ of\ standard\ array)$
$H_{0101} = column\ containing\ 0101 = H_0 + [0101]^t$.

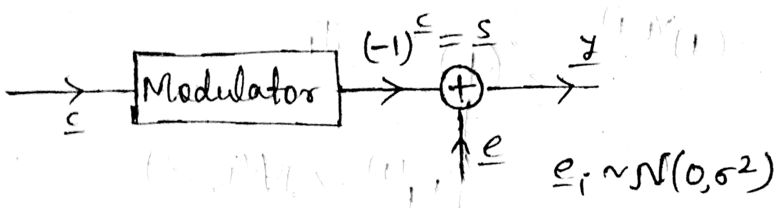
$\vec{H_i}=\vec{H_0}+\vec{C_i}$

Suppose $\vec{0}$ was transmitted and $\vec{e}$ was the error pattern introduced by the BSC. Suppose the decoded codeword is $\vec{C_i}$.Then the residual error often decoding equals?
Next, suppose $\vec{c_j}$ was transmitted and $\vec{e}$ was again the error vector.
Then, $\vec{y}=\vec{c_j}+\vec{e}$
We know that, $\vec{e}\in$H$_i$.
So, $\vec{e}=\vec{c_i}+\vec{a}$, a$\in$E
$\therefore \vec{y}=\vec{c_j}+\vec{c_i}+\vec{a}$
$\implies \vec{y}\in$H$\vec{c_i}+\vec{c_j}$

$\therefore$ the residual error pattern=
$(\vec{c_i}+\vec{c_j})+\vec{c_j}=\vec{c_i}$
(independent of the transmitted codeword)
$\underline{c}+\underline{\hat{c}}=\underline{u}^T G + \underline{\hat{u}}G = (\underline{u}^T+\underline{\hat{u}})G = \underline{\theta}$ (residual error)



Next consider the case of the AWGN channel

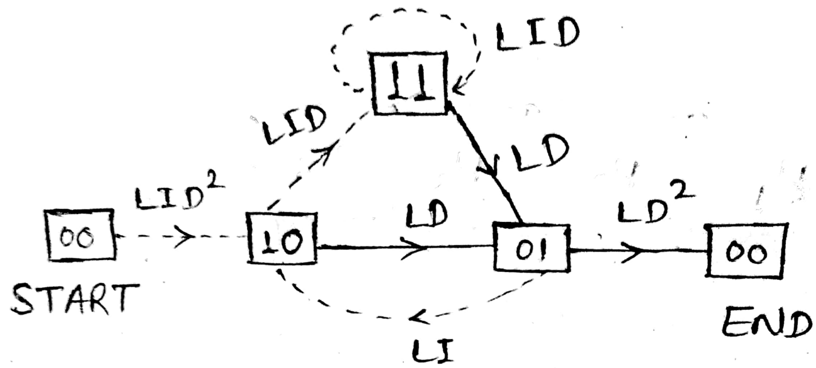


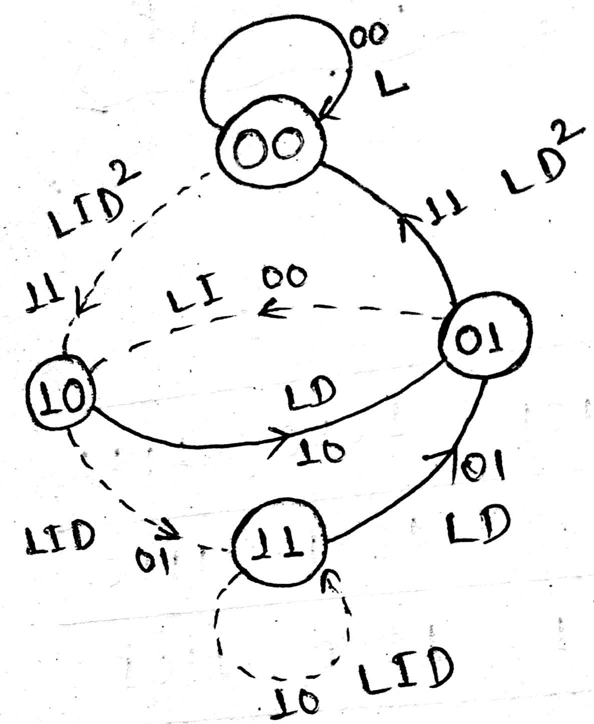
Suppose $\underline{c_i}=0$ was transmitted. $\underline{y}\in H_0$ only.
$\underline{y}^T(-1)^{\underline{0}} \geq \underline{y}^T(-1)^{\underline{c_i}}$, $\underline{c_i}\neq\underline{0}$

Suppose we choose our decision regions such that
$\underline{y}\in H_0 \; iff \; (\underline{y}^T)(-1)^{\underline{0}} \geq (\underline{y}^T)(-1)^{\underline{c_i}} \quad \underline{c_i}\neq\underline{0}$

Next suppose $\underline{y}\in D_i$
$\therefore \underline{y}^T(-1)^{\underline{c_i}} \geq \underline{y}^T(-1)^{\underline{c_j}}$, $\underline{c_j}\neq\underline{c_i}$
$\therefore (\underline{y}\bigodot(-1)^{\underline{c_i}})^T(-1)^{\underline{0}} \geq (\underline{y}\bigodot(-1)^{\underline{c_i}})^T(-1)^{\underline{c_j}+\underline{c_i}}$

Modified State Diagram





$$\begin{bmatrix} A_{10} \\ A_{11} \\ A_{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 & LI \\ LID & LID & 0 \\ LD & LD & 0 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{01} \end{bmatrix} + \begin{bmatrix} LID^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & -LI \\ -LID & 1-LID & 0 \\ -LD & -LD & 1 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{01} \end{bmatrix} = \begin{bmatrix} LID^2 \\ 0 \\ 0 \end{bmatrix}$$

$$Let\ \ T = \begin{bmatrix} 1 & 0 & -LI \\ -LID & 1-LID & 0 \\ -LD & -LD & 1 \end{bmatrix}$$

$det(T) = 1(1-LID) - LI(L^2ID^2 + LD(1-LID))$
$= 1 - LID - L^2ID$
$= 1 - LID(1+L)$

$\therefore$ $\mathrm{A}_{01} = \frac{(LID^2)(LD)}{1-LID(1+L)}$ where $\begin{vmatrix} -LID & 1-LID \\ -LD & -LD \end{vmatrix}$ = LD

$\therefore$ $\mathrm{A}_{END} = \frac{(LID^2)(LD)(LD^2)}{1-LID(1+L)} = \frac{L^3ID^5}{1-LID(1+L)}$
Set L=1, I=1
$\implies \mathrm{A}_{END} = \frac{D^5}{1-2D}$ $= \mathrm{D}^5(1+2D+4D^2+...)$

$\mathrm{A}_{END}(L,I,D) = \sum_{i,j,k} a_{i,j,k} L^i I^j D^k$