E2 205 Error-Control Coding
Lecture 25

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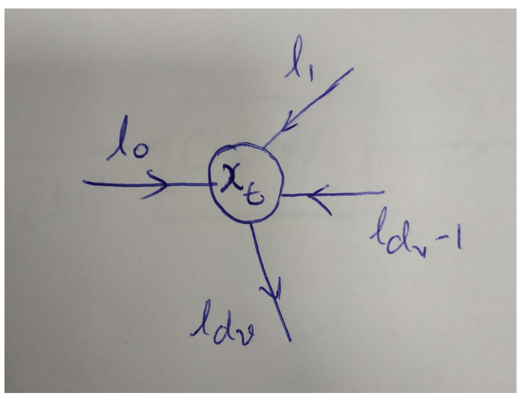
1 LDPC Code: Density Evolution

1.1 BP at a variable node



$$l_{d_v} = \sum_{j=0}^{d_v-1} l_j \tag{1}$$

$l_j = P(x_t/E_j)$
$E_0 = y_t$
$E_i =$ some disjoint subset of received symbols.

1.2 Density Evolution

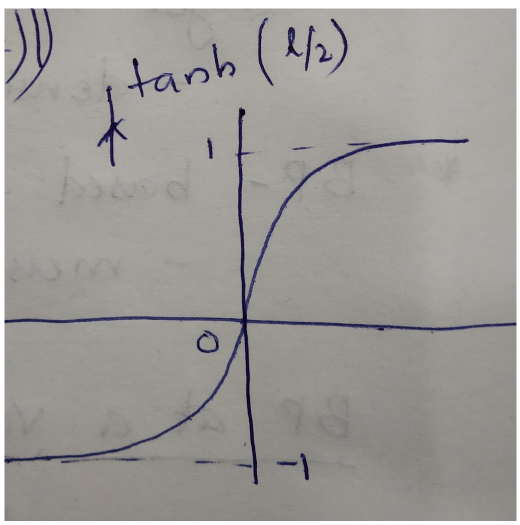
Assume $\underline{1}$ was transmitted. The received symbols are all independent.This makes the $\{l_j\}_{j=0}^{d_v-1}$ independent.
Therefore the pdf of l_{d_v} is the convolution of the pdfs of the l_j , $0 \leqslant j \leqslant d_v - 1$
Therefore,
$P_{l_{d_v}} = \mathbb{F}^{-1}\{\mathbb{F}\{P_{l_0}\} * \mathbb{F}\{P_{l_j}\}^{d_v-1}\}$ for some j, $1 \leqslant j \leqslant d_v - 1$

1.3 BP at a check node

We saw that here message passing takes on the form

$$tanh(\frac{l_{d_v}}{2}) = \prod_{j=1}^{d_c-1} tanh(\frac{l_j}{2}) \tag{2}$$

Would like to take logs on both sides of equation 2.

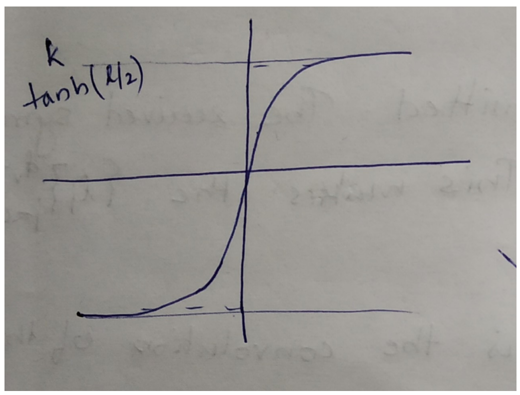


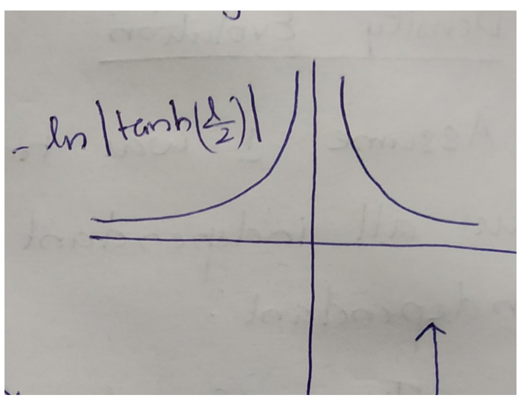
However, $tanh(\frac{l}{2})$ can take negative values. So we replace $tanh(\frac{l}{2})$ by (X, Y) where:
$X = sgn(l)$
$Y = -ln|tanh(\frac{l}{2})|$
and

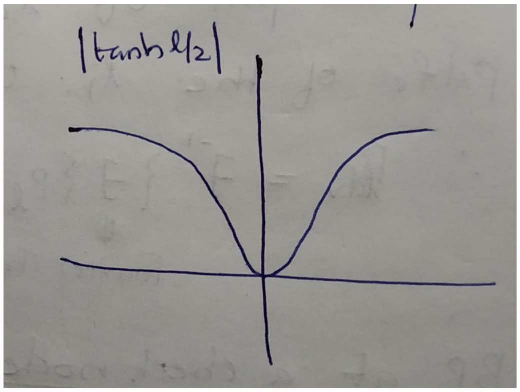
$$sgn(l) = \begin{cases} 0 & \text{for } l \geqslant 0 \\ 1 & \text{for } l < 0 \end{cases}$$

(Think of $(-1)^{sgn(l)}$ as indicating the sign of l)

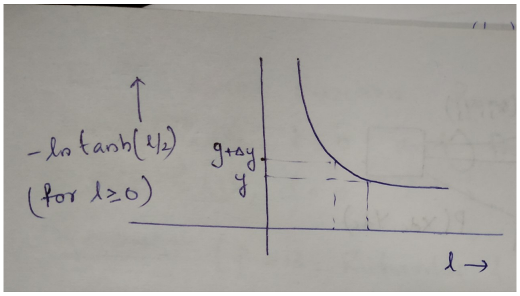
From equation 2 it follows that
$X_{d_c} = \sum_{j=1}^{d_c-1} X_j \, mod \, 2$
$Y_{d_c} = \sum_{j=1}^{d_c-1} Y_j$







Let $P_j(0,y)$ be such that $P_j(0,y+\Delta y) - P_j(0,y)$ is the probability $y \leqslant Y \leqslant y + \Delta y$.
$Y = -ln|\frac{tanh(L)}{2}| = -ln(tanh(\frac{L}{2}))$
$(Y = g(L), L \geqslant 0)$



ASIDE: Change of variables

$Y = g(X)$
$f_Y(y) = \frac{f_X(g^{-1}(y))}{|\frac{dy}{dx}|_{x=g^{-1}(y)}}$

Let $g^{-1}(y) = h(y)$
Then $|\frac{dy}{dx}| = \frac{1}{|\frac{dh}{dy}|}$

$e^{-Y} = \frac{e^L - 1}{e^L + 1}$
$e^L(1 - e^{-Y}) = e^{-Y} + 1$
$e^L = \frac{1 + e^{-Y}}{1 - e^{-Y}}$
$e^{-L} = \frac{1 - e^{-Y}}{1 + e^{-Y}}$
$= \frac{e^Y - 1}{e^Y + 1}$
Therefore,

$$L = -ln\, tanh(\frac{Y}{2})$$
$$= h(Y)$$

$$|\frac{dh}{dy}| = \frac{1}{tanh\frac{y}{2}} \frac{d}{dy}(\frac{e^y - 1}{e^y + 1})$$
$$= \frac{1}{tanh\frac{y}{2}} \frac{(e^y + 1)e^y - (e^y - 1)e^y}{(e^y + 1)^2}$$
$$= \frac{e^y + 1}{e^y - 1} * \frac{2e^y}{(e^y + 1)^2}$$
$$= \frac{2e^y}{e^y - 1}$$
$$= \frac{1}{sinh(y)}$$

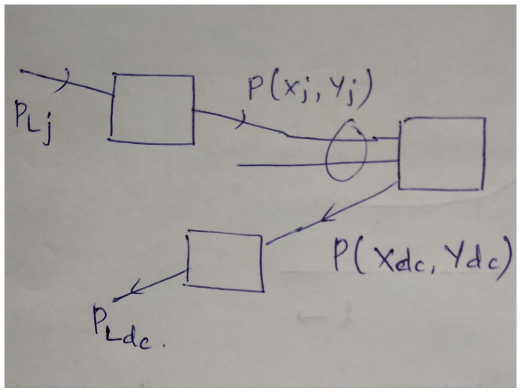
Therefore,

$$P(0, y) = \frac{1}{sinh(y)} P_L(-ln\, tanh(\frac{y}{2}))$$

Similarly,

$$P(1, y) = \frac{1}{sinh(y)} P_L(ln\, tanh(\frac{y}{2}))$$
$$sinh(y) = \frac{e^y - e^{-y}}{2}$$

In this way one can find $P_j(0,y),\ 1 \leqslant j \leqslant d_c - 1$
Define a transform on P(x,y)



$$\mathbb{E}[(-1)^{\lambda X} e^{-sY}] = \sum_{x=0}^{1} (-1)^{\lambda x} \int_0^{\infty} e^{-sy} P(x,y) dy \tag{3}$$

$$\begin{aligned}
\mathbb{E}[(-1)^{\lambda X_{d_c}} e^{-sY_{d_c}}] &= \mathbb{E}[\prod_{j=1}^{d_c-1} (-1)^{\lambda X_j} e^{-sY_j}] \\
&= \prod_{j=1}^{d_c-1} \mathbb{E}[(-1)^{\lambda X_j} e^{-sY_j}] \\
&= \prod_{j=1}^{d_c-1} [\hat{P}_j(0,s) + (-1)^{\lambda} \hat{P}_j(1,s)]
\end{aligned}$$

where,
$X_{d_c} = \sum X_j \, mod \, 2$
$Y_{d_c} = \sum_j Y_j$

Setting $\lambda = 0$ we get

$$\hat{P_{d_c}}(0,s) + \hat{P_{d_c}}(1,s) = \prod_{j=1}^{d_c-1} [\hat{P_j}(0,s) + \hat{P_j}(1,s)]$$

$$\hat{P_{d_c}}(0,s) - \hat{P_{d_c}}(1,s) = \prod_{j=1}^{d_c-1} [\hat{P_j}(0,s) - \hat{P_j}(1,s)]$$

Thus we can recover $\hat{P_{d_c}}(0,s)$ and $\hat{P_{d_c}}(1,s)$
In the reverse direction, from $P(0,y)$ we can recover $P_L(l), l \geqslant 0$ and from $P(1,y)$ we can recover $P_L(l), l \leqslant 0$

Theorem: Over the probability space of all graphs $C(d_v, d_c)$ and channel realizations, let Z be the number of incorrect messages among all n_{d_v} variable to check node message passed at iteration l. Let p be the expected number of incorrect messages passed along an edge with a tree-like directed neighborhood of depth at least $2l$ at the l^{th} iteration. Then there exists positive constants $\beta = \beta(d_v, d_c, l)$ and $\gamma = \gamma(d_c, d_v, l)$ such that:

i Concentration around the expected value.
For any $t > 0$ we have:

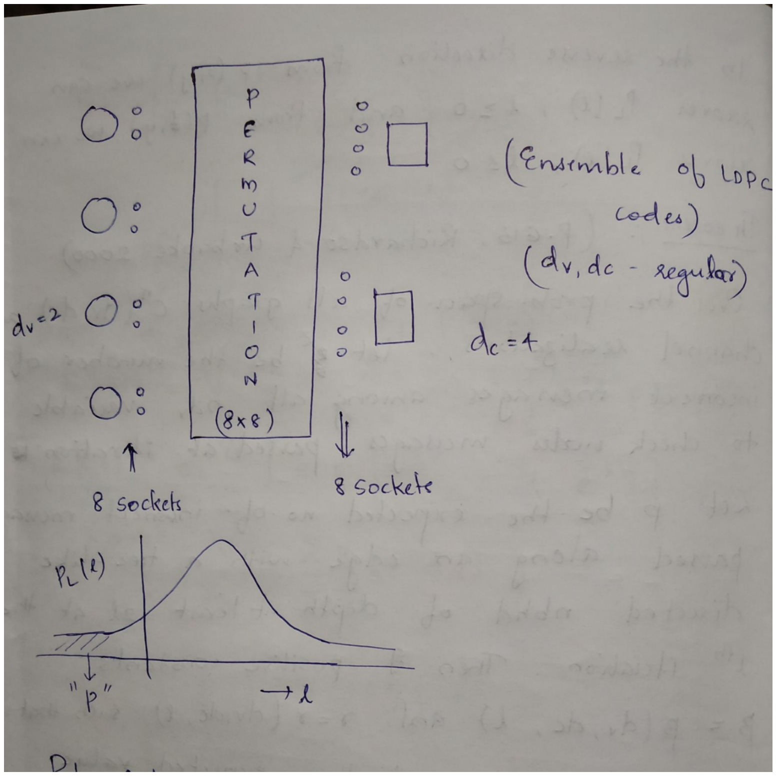
$$Pr\{|Z - \mathbb{E}[Z]| > \frac{nd_vt}{2}\} \leqslant 2e^{-\beta t^2 n}$$

ii Convergence to the cycle-free case:
For any $t > 0$ and $n > \frac{2\gamma}{t}$, we have

$$|\mathbb{E}[Z] - nd_vp| < \frac{nd_vt}{2}$$

iii Concentration around the cycle-free case:
For any $t > 0$ and $n > \frac{2\gamma}{t}$, we have

$$Pr\{|Z - nd_vp| > \frac{nd_vt}{2}\} \leqslant 2e^{-\beta t^2 n}$$

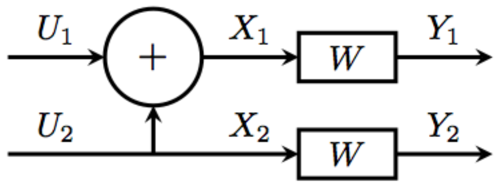


2 Polar Codes

A coding scheme (polar coding) is presented that achieve the "symmetric capacity"(capacity assuming both inputs are equally likely) of the BI-DMC(Binary Inout Discrete Memoryless Channel).

Equally likely inputs achieve capacity on large class of channels: BSC, BEC and others.

2.1 Basic Idea



$$\begin{aligned} I(U_1U_2;Y_1Y_2) &= I(X_1X_2;Y_1Y_2) \\ &= 2I(W) \end{aligned}$$

$$\begin{aligned} I(X_1X_2;Y_1Y_2) &= H(Y_1Y_2) - H(Y_1Y_2/X_1X_2) \\ &= H(Y_1) + H(Y_2) - H(Y_1/X_1X_2) - H(Y_2/X_1X_2Y_1) \\ &= H(Y_1) + H(Y_2) - H(Y_1/X_1) - H(Y_2/X_2) \\ &= I(X_1;Y_1) + I(X_2;Y_2) \\ &= 2I(W) \end{aligned}$$

$$\begin{aligned} I(U_1U_2;Y_1Y_2) &= I(U_1;Y_1Y_2) + I(U_2;Y_1Y_2/U_1) \\ &= I(Y_1Y_2;U_1) + I(Y_1Y_2U_1;U_2) \\ &= I(W^-) + I(W^+) \end{aligned}$$