E2 205 Error-Control Coding
Lecture 26

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1 Polar Code

1.1 Channel Combining

Assumptions:

- Considering only Binary Input Discrete Memoryless Channels (BI-DMC)
- Throughout the inputs are equally likely

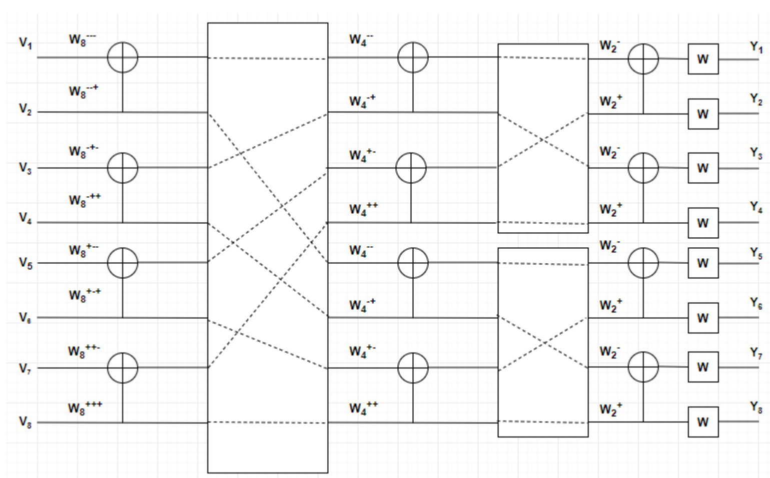

Figure 1: Channel Combining

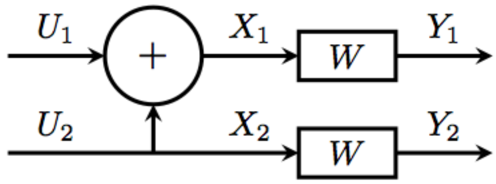

Figure 2:

From previous lecture we know that

$$\begin{aligned} I(U_1U_2;Y_1Y_2) &= I(X_1X_2;Y_1Y_2) \\ &= I(X_1;Y_1) + I(X_2;Y_2) \\ &= 2I(W) \\ Also, \end{aligned}$$

$$\begin{aligned} I(U_1;Y_1Y_2) + I(U_2;Y_1Y_2|U_1) &= I(Y_1Y_2;U_1) \\ &= I(Y_1Y_2U_1;U_2) \\ &= I(W^-) + I(W^+) \end{aligned}$$

$$\begin{aligned} W^{(1)}(y_1^2|u_1) &= \frac{1}{2}\sum_{u_2} W(y_1|u_1+u_2)W(y_2|u_2) \\ W^{(2)}(y_1^2u_1|u_2) &= \frac{1}{2}W(y_1|u_1+u_2)W(y_2|u_2) \end{aligned}$$

$$\begin{aligned} W^{(1)} &\equiv W^0 \equiv W^- \\ W^{(2)} &\equiv W^1 \equiv W^+ \end{aligned}$$

Let us define

$$W_N^{(i)}(y_1^m u_1^{i-1}|u_i) \equiv \frac{1}{2^{N-1}} \sum P(y_1^N|u_1^N)$$

These are called the marginal channels(the i^{th} marginal channels)

$$
\begin{aligned}
P(y_1^N u_1^{i-1}|u_i) &= 2P(y_1^N u_1^i) \\
&= 2\sum_{u_{i+1}^N} P(y_1^N u_1^N) \\
&= \frac{1}{2^{N-1}} \sum_{u_{i+1}^N} P(y_1^N|u_1^N)
\end{aligned}
$$

Assuming all u_i are equally likely.

We have the following recursion:

$$
N = 2^n (say)
$$

$$
W_N^{(2i-1)}(y_1^N u_1^{2i-2}|u_{2i-1}) \equiv \frac{1}{2^{N-1}} \sum_{u_{2i}^N} P(y_1^N|u_1^N)
$$

$$
u_{2i}^N = \{u_{2i}, u_{2i+1}, ...u_N\}
$$

$$
= \frac{1}{2}\sum_{u_{2i}}\{\frac{1}{2^{\frac{N}{2}-1}} \sum_{u_{2i+1,o}^N + u_{2i+1,e}^N} P(y_1^{\frac{N}{2}}|u_{i,o}^N + u_{i,e}^N)\frac{1}{2^{\frac{N}{2}-1}} \sum_{u_{2i+1,e}^N} P(y_{\frac{N}{2}+1}^N|u_{i,e}^N)\}
$$

$$
let \quad V_i = u_{2i-1} + u_{2i} \quad X_i = u_2 i \quad 1 \leq i \leq \frac{N}{2}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{u_{2i}}\{\frac{1}{2^{\frac{N}{2}-1}} \sum_{V_{i+1}^{\frac{N}{2}}} P(y_1^{\frac{N}{2}}|V_1^{\frac{N}{2}})\frac{1}{2^{\frac{N}{2}-1}} \sum_{X_{i+1}^{\frac{N}{2}}} P(y_{\frac{N}{2}+1}^N|X_1^{\frac{N}{2}})\} \\
&= \frac{1}{2}\sum_{u_{2i}} W_{\frac{N}{2}}^{(i)}(y_1^{\frac{N}{2}} V_1^{i-1}|V_i) W_{\frac{N}{2}}^{(i)}(y_{\frac{N}{2}+1}^N X_1^{i-1}|X_i) \\
&= \frac{1}{2}\sum_{u_{2i}} W_{\frac{N}{2}}^{(i)}(y_1^{\frac{N}{2}} u_{1,o}^{2i-2} + y_{1,e}^{2i-2}|u_{2i-1} + u_{2i}) W_{\frac{N}{2}}^{(i)}(y_{\frac{N}{2}+1}^N y_{1,e}^{2i-2}|u_{2i})
\end{aligned}
$$

1.2 Fundamental Channel Recursion

$$P(y_1^N|u_1^N) = \underbrace{P(y_1^{\frac{N}{2}}|u_{1,o}^N + y_{1,e}^N)}_{W_{N/2}\text{ Channel}}\underbrace{P(y_{\frac{N}{2}+1}^N|u_{1,e}^N)}_{W_{N/2}\text{ Channel}}$$

Similarily, one can show that,
$W_N^{2i}(y_1^N y_1^{2i-1}|u_{2i}) = \frac{1}{2}W_{\frac{N}{2}}(y_1^{\frac{N}{2}} u_{1,o}^{2i-2} + u_{1,e}^{2i-2}|u_{2i-1} + u_{2i})W_{\frac{N}{2}}(y_{\frac{N}{2}+1}^N u_{1,e}^{2i-2}|u_{2i})$

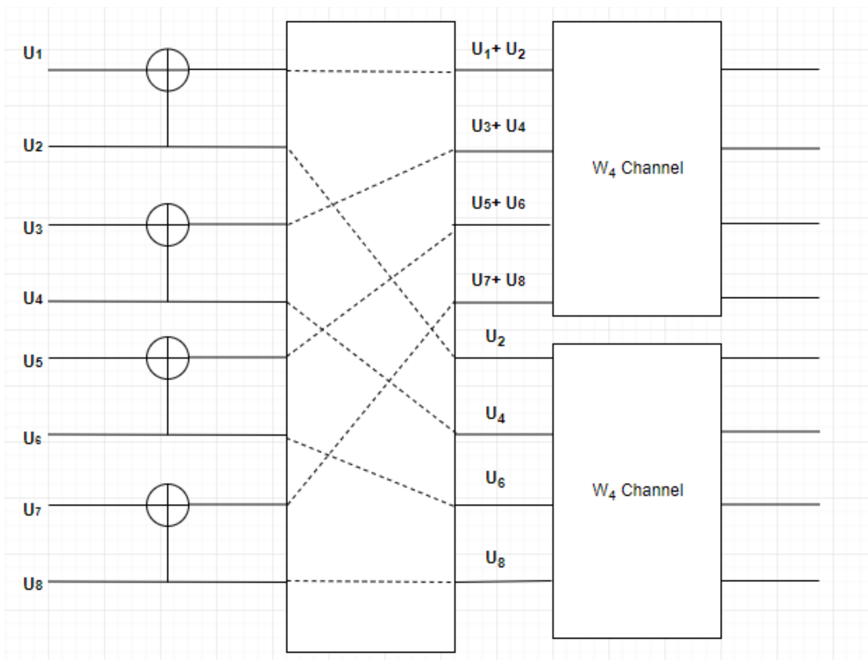

Figure 3: