Page No.
Date / /20

Lec 13 - Shannon - Fano - Elias Coding

27th Sep 2017

Recap
- Huffman Codes
- examples
- formal derivation

Today
- Shannon - Fano - Elias Coding
- examples
- Competitive optimality

SHANNON - FANO - ELIAS CODING -

Setting $|\mathcal{X}| = m$. Assume $\mathcal{X} = \{1, \ldots, m\}$

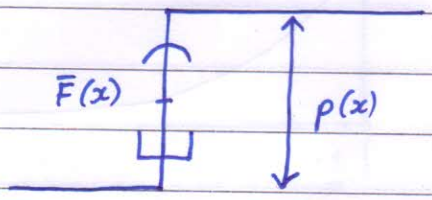
$p(x)$ - pmf

$$F_X(x) = \sum_{a \le x} p(a)$$

Define,

$$\bar{F}(x) = \sum_{a < x} p(a) + \frac{1}{2} p(x)$$

Note that assuming $(p(x) > 0 \ \forall x \in \mathcal{X})$, $\bar{F}(x)$ is well defined. given $F(x)$ we can uniquely recover x.



Suppose $\bar{F}(x) = 0.y_1 y_2 \ldots y_{l(x)} \ldots$

We want to pick $l(x)$ s.t.

$$\left[\lfloor \bar{F}(x) \rfloor_{l(x)}, \lfloor \bar{F}(x) \rfloor_{l(x)} + \frac{1}{2^{l(x)}} \right) \subseteq \left[F(x) - p(x), F(x) \right)$$

$\therefore$ want,

$$\frac{1}{2^{l}} < \frac{p(x)}{2} \Rightarrow 2^{l(x)-1} > \frac{1}{p(x)}$$

$\therefore \; l(x) - 1 > -\log_2 p(x)$

$\therefore \; l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil + 1$, will suffice

Encode as,

$$x \longmapsto y_1 \ldots y_{l(x)}$$

Then this code is a prefix code since the intervals,

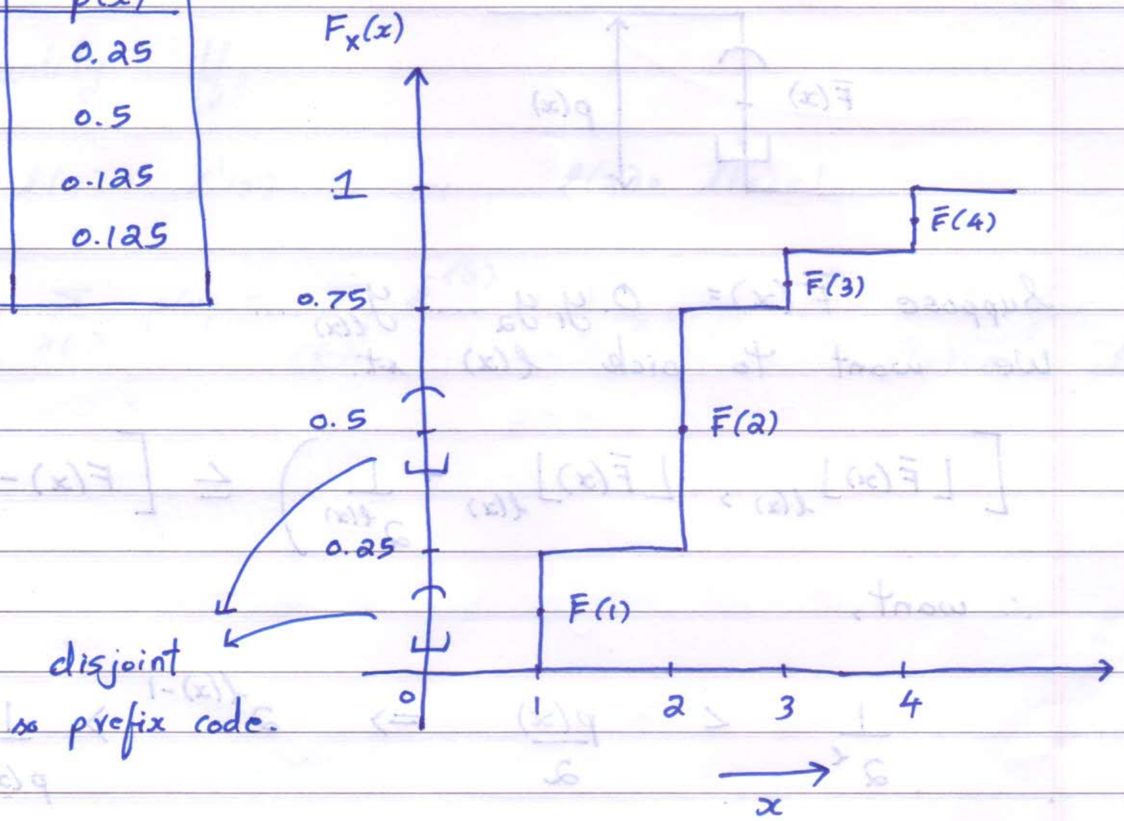
$$\left[\lfloor \bar{F}(x) \rfloor_{l(x)} , \lfloor \bar{F}(x) \rfloor_{l(x)} + \frac{1}{2^{l(x)}} \right)$$ are disjoint.

Note - $\lfloor \bar{F}(x) \rfloor_{l(x)} = \lfloor 0.y_1 \ldots y_i \ldots \rfloor_{l(x)}$

$= 0.y_1 y_2 \ldots y_{l(x)}$

example -

x	$p(x)$
1	0.25
2	0.5
3	0.125
4	0.125



Page No.
Date / /20

Example-

$0.y_1y_2 - y_{l(x)}$	x	$p(x)$	$F(x)$	$\bar{F}(x)$	$l(x)$	Code, $c(x)$
0.001	1	0.25	0.25	$0+\frac{1}{8}$	3	001
0.1	2	0.5	0.75	$\frac{1}{4}+\frac{1}{4}$	2	10
0.1101	3	0.125	0.875	$\frac{3}{4}+\frac{1}{16}$	4	1101
0.1111	4	0.125	1	$\frac{7}{8}+\frac{1}{16}$	4	1111

Average length, $L(\mathcal{C}) = \frac{3}{4} + \frac{2}{2} + \frac{4}{8} + \frac{4}{8}$

$= 2.75$ bits

$H(x) = \frac{2}{4} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8}$

$= 1.75$ bits

Note that under SFE Coding, we have

$$L(\mathcal{C}) = \sum_{x \in \mathcal{X}} p(x)\, l(x)$$

$$= \sum_{x \in \mathcal{X}} p(x) \left(\left\lceil \log \frac{1}{p(x)} \right\rceil + 1 \right)$$

$$\cancel{4} < \sum_{x \in \mathcal{X}} p(x) \left(\log \frac{1}{p(x)} + 2 \right)$$

$$= H(x) + 2 \quad (\therefore \text{ within } 2 \text{ bits})$$

(See text for more examples).

Theorem (Comparison with Shannon Code)

Let $\{l(x)\}$ be the lengths associated with the Shanon Code and $\{l'(x)\}$ be lengths associated with any other ~~code.~~ uniquely decodable code.

Then,

$$P_r\left(l(x) \geq l'(x) + c\right) \leq \frac{1}{2^{c-1}}$$

Proof-

$$P_r\left(l(x) \geq l'(x) + c\right)$$

$$= \sum_{\substack{x \in \mathcal{X}_0 \\ l(x) \geq l'(x)+c}} p(x)$$

$$l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil$$

$$< \log \frac{1}{p(x)} + 1$$

$$\therefore \; p(x) < \frac{1}{2^{l(x)-1}}$$

$$< \sum_{\substack{x \in \mathcal{X}_0 \\ l(x) \geq l'(x)+c}} \frac{1}{2^{l(x)-1}}$$

$$\leq \sum_{x \in \mathcal{X}_0} \frac{1}{2^{l'(x)+c-1}}$$

$$= \frac{1}{2^{c-1}} \sum_{x \in \mathcal{X}_0} 2^{-l'(x)}$$

$$\leq \frac{1}{2^{c-1}} \quad \text{(because of KI)}$$

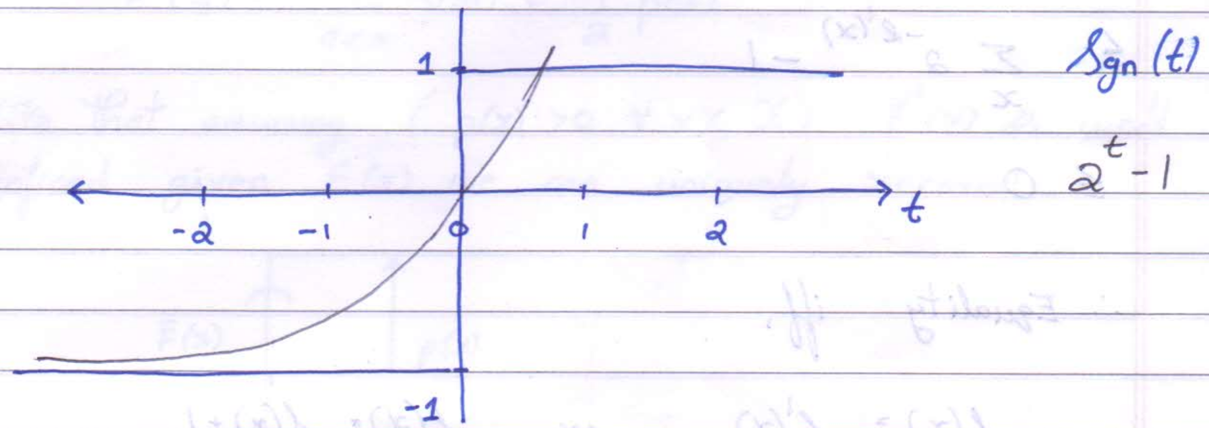
Theorem-

For a d-adic pmf $p(x)$, let $\left\{ l(x) = \log \frac{1}{p(x)} \right\}$ be the word lengths of the binary Shannon Code, and $\{ l'(x) \}$ be the lengths of any other uniquely decodable code,
Then,

$$P_r \left(l(x) < l'(x) \right) \overset{\leq}{\cancel{\geq}} P_r \left(l'(x) > l(x) \right)$$

with equality iff $l(x) = l'(x) \ \forall \ x \in \mathcal{X}_o$.
Thus the code length assignment $l(x) = \log \frac{1}{p(x)}$ is Competitively optimal.

Proof-



Define,

$$Sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \quad \text{(the sign function)}$$

Note that at integer values of the argument,

$$2^{t-1} \geq sgn(t)$$

Consider,

$$Pr(l(x) > l'(x)) - Pr(l'(x) > l(x))$$

$$= \sum_{\substack{x \in \mathcal{X}. \\ l(x) > l'(x)}} p(x) - \sum_{\substack{x \\ l'(x) > l(x)}} P(x)$$

$$= \sum_x p(x)\, Sgn(l(x) - l'(x))$$

$$\leq \sum_x p(x)\left(2^{(l(x) - l'(x))} - 1\right)$$

$$= \sum_x 2^{-l(x)}\left(2^{(l(x) - l'(x))} - 1\right)$$

$$\leq \sum_x 2^{-l'(x)} - 1$$

$$\leq 0$$

$\therefore$ Equality iff,

$$l(x) = l'(x) \quad \text{or} \quad l(x) = l'(x) + 1$$

But, $\sum_{x \in \mathcal{X}} p(x) = \sum_{x \in \mathcal{X}} 2^{-l(x)} = 1$, and the KI for $l(x) = l'(x)$ all x.