Lec14: Arithmetic Coding
Channel Capacity

DATE 9 · 10 · 17

<u>Recap:</u>

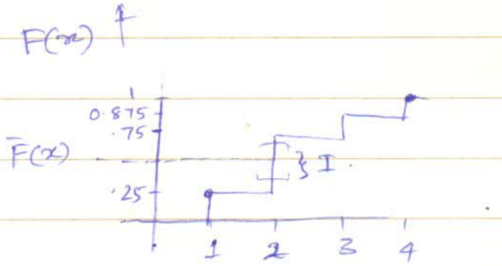
Data compression

- R·I
- $H(X) \le L^*(X) < H(X)+1$
- Huffmann code
- Shannon-Fano-Elias code

$L(c) < H(c)+2$

- competitive optimality of shannon coding $\lceil \log \frac{1}{p(x)} \rceil$

Shannon-Fano-Elias Coding



SF-E-Code.

Given $x \in \mathcal{X}_0$, compute $\bar{F}(x)$.

$$\bar{F}(x) = p(X < x) + \frac{p(x)}{2}$$

$$F(x) = P(X \le x)$$

Expand say in binary

$$\bar{F}(x) = 0.y_1 y_2 \ldots y_\ell y_{\ell+1} \ldots$$

Consider the interval $[0.y_1 y_2 \ldots y_\ell,\ 0.y_1 y_2 \ldots y_\ell 111\ldots) = I$.

length $= \frac{1}{2^\ell}$. Want to ensure that

$$I \subseteq [F(x)-p(x),\ F(x))$$

Ensure this by selecting

Say $p(x) = 0.0128$
$\lceil \log \frac{1}{p(x)} \rceil = \lceil 7.8 \rceil = 8$
$\therefore$ use $\ell(x) = 8+1 = 9$.
$\bar{F} = 0.3264 = 0.\boxed{010100111}$
to 9 digits (the binary encoder o/p)

$$\frac{1}{2^\ell} < \frac{p(x)}{2} \qquad 2^\ell > \frac{2}{p(x)}$$

$$\therefore \ell - 1 > \log \frac{1}{p(x)} \implies \ell > \log \frac{1}{p(x)} + 1$$

The choice

$$\ell = \left\lceil \log \frac{1}{p(x)} \right\rceil + 2 \text{ will suffice.}$$

~~If~~ It follows that $L_{SFE}(c) < H(X)+2$ bits.

<u>Arithmetic coding</u>: Employs SFE encoding

$\boxed{\text{Source}} \xrightarrow{x^n \in \mathcal{X}^n}$

Suppose we wish to compress $\{X_k\}_{k=1}^{n}$

WOLOG assume $\mathcal{X}_0 = \{0, 1, \ldots |\mathcal{X}|-1\}$

<u>Step 1:</u> Form $X = 0.X_1 X_2 \ldots X_n$. We now look to

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encode X using $\bar{F}_X(x)$

$$F_X(0.x_1x_2\ldots x_n) = \Pr(0.X_1X_2\ldots X_n \le 0.x_1x_2\ldots x_n)$$

$$= \Pr(X_1 < x_1) + \Pr(X_1 = x_1, X_2 < x_2)$$

$$+ \ldots \Pr(X_1 = x_1, X_2 = x_2, \ldots X_{n-1} < x_{n-1})$$

$$+ \Pr((X_1 \ldots X_n) = (x_1, \ldots x_n), X_n \le x_n)$$

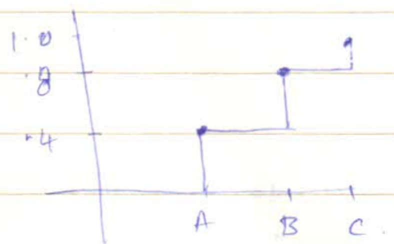
(Assume binary below)

$$F_X^{(2)}(0.x_1x_2\ldots x_n)$$

$$= \sum_{k=1}^{n} p(x^{k-1}0)\, x_k = \sum_{k=1}^{n} p(x^{k-1})\, x_k\, P(X_k = 0)$$

Eg: $|\mathcal{X}| = 3$, $\mathcal{X} = \{A, B, C\} = \{0, 1, 2\}$ (say ordering)

$P_X(A) = 0.4$, $P_X(B) = 0.4$, $P_X(C) = 0.2$



Goal is to encode ACAA

$$F_l(0.ACAA)$$

$$= \Pr(x_1 < A) + \Pr(x_1 = A, x_2 < C)$$

$$+ \Pr(x_1 = A, x_2 = C, x_3 < A)$$

$$+ \Pr(x_1 = A, x_2 = C, x_3 = A, x_4 < A)$$

$$= 0 + 0.4(0.8) + 0 + 0 = 0.32$$

$F_h(0.ACAA)$ = same except for the last term

$$P(x_1 = A, x_2 = C, x_3 = A, x_4 \le A)$$

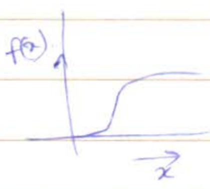
$\parallel$

0.3328

$$= 0.4 \times 0.2 \times 0.4 \times 0.4 = 0.0128$$

$$\bar{F}(x) = \underline{0.3264}$$

Motivation



$$U = F_X(X) \in [0, 1]$$

$$\Pr(U \le u) = \Pr\{F_X(X) \le u\}$$

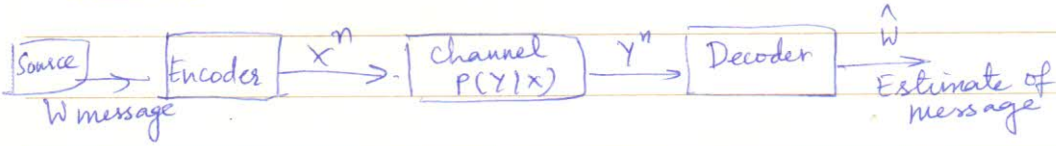
$$= \Pr\{X \le F_X^{-1}(u)\}$$

$$= F_X\{F_X^{-1}(u)\} = u$$

Natural

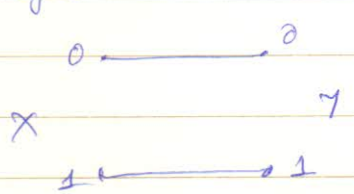
Channel capacity

Setting



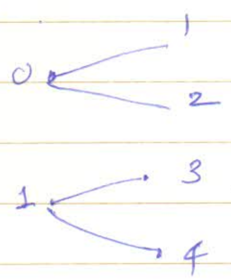
$$\underset{\text{information capacity}}{C} = \max_{p(x)} I(X;Y)$$

Eg 1:



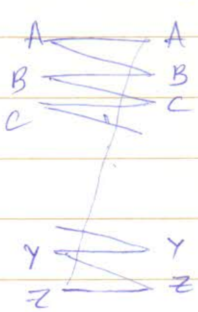
$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(Y) - \underbrace{H(Y|X)}_{0}$$
$$= H(Y)$$
$$C = \max_{p(x)} I(X;Y) = 1 \text{ bit}$$

Eg: 2



$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(X)$$
$$\Rightarrow C = 1 \text{ bit}$$

Eg: 3 (noisy typewriter)



$$I(X;Y)$$
$$= H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X)$$
$$\sum p(x) H(Y|X=x)$$
$$= \sum p(x) \cdot 1 = 1$$

$$C = \max_{p(x)} H(Y) - 1 = \underline{\log 26 - 1} = \log_2 13 \text{ bits}$$

Scheme to achieve this is to only consider A, C, E, ...