Final Exam: 26th Nov (Sunday)
9am - 11am

Lec 20: Differential entropy

DATE 03 · 11 · 17

Let X be a continuos rv having probability density function (pdf) $f_X(x) := f(x)$

Let

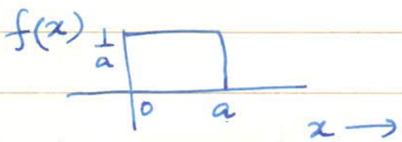
$S = \{x \mid f(x) > 0\}$ → support of $f(x)$

Defn: The differential entropy $h(X)$ of X is defined by

$$\boxed{h(X) = + \int_S f(x) \log \frac{1}{f(x)}\, dx} = E[-\log f(X)] \quad \to (1)$$

(We will assume that every integral encountered exists) (note that $h(X)$ is really a function of $f(x)$)

Eg: (1)



Let X be uniform over $[0, a]$. Then

$$h(X) = -\int_S f(x) \log f(x)\, dx = -\int_0^a \frac{1}{a} \log \frac{1}{a}\, dx$$

$$= \log(a) \text{ bits} \quad -(2)$$

Note that if $a < 1$, then $h(X) < 0$ (Thus differential entropy can be negative)

Eg: (2)

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{x^2}{2\sigma^2}}$$

$$h(X) = E[-\log f(X)] = E\left[-\log \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{X^2}{2\sigma^2}}\right]$$

$$= \frac{1}{2}\log 2\pi\sigma^2 + E\left\{\frac{X^2}{2\sigma^2}\right\} \log e = \frac{1}{2}\log 2\pi e\sigma^2 \quad \to (3)$$

Natural

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AEP for continuos sources $\qquad$ Assume $\{X_n\}$ iid $\sim f(x)$

Let X be as above (continuos)

$$A_\epsilon^{(n)} = \left\{ \underline{x} \in S^n \;\middle|\; \left| h(x_1) - f(x) \log \frac{1}{f(x)} \right| < \epsilon \right\} \quad \to (4)$$

Note that

$$\frac{1}{n} \log \frac{1}{f(\underline{x})} = \frac{1}{n} \log \frac{1}{\prod_{i=1}^{n} f(x_i)} = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\left[\log \frac{1}{f(X_i)} \right]}_{Y_i \text{ (say)}}$$

Note $E[Y_i] = \int_S f(x) \log \frac{1}{f(x)}\, dx = h(x_1)$

Thus $\quad \frac{1}{n} \log \frac{1}{f(\underline{x})} \xrightarrow[(WLLN)]{} h(x_1)$ in probability

and hence

$$P_r \left\{ \left| h(x_1) - \frac{1}{n} \log \frac{1}{f(\underline{x})} \right| \geq \epsilon \right\} \to 0 \text{ as } n \to \infty$$

Thus given any $\epsilon > 0$, $\exists\, n_\epsilon$ st $\forall\, n \geq n_\epsilon$,

$$P_r \left\{ \left| h(x_1) - \frac{1}{n} \log \frac{1}{f(\underline{x})} \right| \geq \epsilon \right\} \leq \epsilon$$

Note that $\underline{x} \in A_\epsilon^{(n)} \Rightarrow$

$$\left| h(x_1) - \frac{1}{n} \log \frac{1}{f(\underline{x})} \right| < \epsilon$$

$$\Rightarrow h(x_1) - \epsilon < \frac{1}{n} \log \frac{1}{f(\underline{x})} < h(x_1) + \epsilon$$

$$\Rightarrow 2^{h(x_1) - \epsilon} < \frac{1}{f(\underline{x})} < 2^{n(h(x_1) + \epsilon)}$$

$$2^{-(h(x_1)+\epsilon)} < f(\underline{x}) < 2^{-n(h(x_1) - \epsilon)} \quad - (5)$$

~~Thm~~ Defn: The Volume ($Vol(A)$), $A \subseteq R^n$ is defined as

$$Vol(A) = \int_A dx_1 \ldots dx_n$$

Thm a) $Pr(A_\epsilon^{(n)}) \geq (1-\epsilon)$ for n sufficiently large

b) $Vol(A_\epsilon^{(n)}) \leq 2^{n(h(x)+\epsilon)}$

c) $Vol(A_\epsilon^{(n)}) \geq (1-\epsilon)\, 2^{n(h(x)-\epsilon)}$

Pf: (a) follows from WLLN

$$Pr\left\{ \left| h(x_1) - \log \frac{1}{f(\underline{x})} \right| \geq \epsilon \right\} \leq \epsilon$$

$$\Rightarrow Pr\left\{ \left| h(x_1) - \log \frac{1}{f(\underline{x})} \right| < \epsilon \right\} = Pr(A_\epsilon^{(n)}) \geq 1-\epsilon$$

(b) $$1 \geq \int_{A_\epsilon^{(n)}} f(\underline{x})\, d\underline{x} > \int_{A_\epsilon^{(n)}} 2^{-n(h(x_1)+\epsilon)}\, d\underline{x}$$

$$= 2^{-n(h(x_1)+\epsilon)}\, Vol(A_\epsilon^{(n)})$$

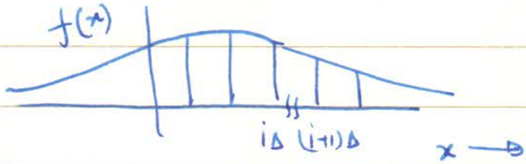
$$\Rightarrow Vol(A_\epsilon^{(n)}) \leq 2^{n(h(x_1)+\epsilon)}$$

(c) Similarly for n sufficiently large

$$1-\epsilon \leq P(A_\epsilon^{(n)}) = \int_{A_\epsilon^{(n)}} f(\underline{x})\, d\underline{x} < 2^{-n(h(x_1)-\epsilon)} \int_{A_\epsilon^{(n)}} d\underline{x}$$

$$\Rightarrow Vol(A_\epsilon^{(n)}) \geq (1-\epsilon)\, 2^{n(h(x_1)-\epsilon)}$$

Quantization



Let us quantize the values taken over by the continuos RV X into bins, with the ith bin

$$[i\Delta, (i+1)\Delta]$$

By the Mean Value Theorem (MVT) $\exists\, x_i \in [i\Delta, (i+1)\Delta]$ st

$$f(x_i)\,\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)\,dx$$

let X^Δ be discrete RV that takes on values

$$X^\Delta = x_i \quad \text{with probability} \quad \underbrace{f(x_i)\,\Delta}_{p_i}$$

Then X^Δ is the quantization of X

Qn: How is $H(X^\Delta)$ related to $h(X)$?

$$H(X^\Delta) := + \sum_{i \in N} \Delta f(x_i) \log \frac{1}{\Delta f(x_i)}$$

$$= + \sum_{i \in N} \Delta f(x_i) \log \frac{1}{\Delta} + \Delta \sum_i f(x_i) \log \frac{1}{f(x_i)}$$

$$= \underbrace{\sum_i p_i \log \frac{1}{\Delta}}_{-\log \Delta} + \Delta \underbrace{\sum_i f(x_i) \log \frac{1}{f(x_i)}}_{\text{as } \Delta \to 0}$$

if $\int f(x) \log \frac{1}{f(x)}\,dx$ is Reimann-integrable, this converges to $h(X)$

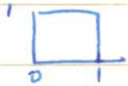
$$\therefore \quad H(X^\Delta) + \log \Delta \to h(X)$$

as $\Delta \to 0$.

Thm: If the density function $f(x)$ of the continuous RV X is Reimann integrable, then $H(X^{\Delta}) + \log \Delta \xrightarrow{\text{s.t. } -f(x)\log f(x) \text{ is}} h(X)$ as $\Delta \to 0$

(thus the entropy of an n-bit quantization (ie., $\Delta = 2^{-n}$) is approximately $h(X) + n$ bits)

Eg (a) Suppose $X \sim$



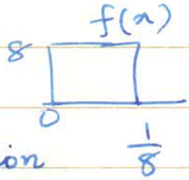
$(a = 1)$

then $h(X) = \log(a) = 0$

then entropy of n bit quantization $\approx 0 + n$
$= n$ bits

(do you agree?)

(b) Suppose $a = \frac{1}{8}$.



$h(X) = -3$ (bits)

Entropy of n bit quantization $\approx n - 3$.

(c) Suppose $X \sim N(0, \sigma^2)$

$\sigma^2 = 100$, then entropy of an n-bit quantization is given

$$h(X) + n \approx n + \frac{1}{2}\log 2\pi e \sigma^2$$
$$\approx n + 5.37 \text{ bits}$$

Joint and Conditional differential entropy

Let $\underline{X} = (X_1, X_2, \ldots X_n)$ have joint density function $f(x_1, x_2, \ldots x_n)$. Then we define the joint differential entropy by

$$h(\underline{X}) = \int_S f(\underline{x}) \log \frac{1}{f(\underline{x})} \, d\underline{x}$$

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Let (X, Y) be continuous RVs having joint density function $f(x, y)$

$$h(X, Y) = \int_S f(x, y) \log \frac{1}{f(x,y)} \, dx \, dy$$

$$h(Y|X) = \int_S f(x, y) \log \frac{1}{f(y|x)} \, dx \, dy$$

expectation for conditional differential entropy

$$= \int_S f(x, y) \log \frac{f(x)}{f(x, y)} \, dx \, dy$$

$$= h(X, Y) - h(X)$$

Joint differential entropy - Guassian case

$\underline{X} \sim \eta(\underline{\mu}, \underline{K})$. $\quad K = E\left[\underbrace{(\underline{X} - \underline{\mu})}_{n\times 1} \underbrace{(\underline{X} - \underline{\mu})^T}_{1\times n}\right]$

$$h(\underline{X}) = E\left\{ \log \frac{1}{f(\underline{X})} \right\} \qquad f(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |K|}} e^{-\frac{(x-\mu)^T K^{-1} (x - \mu)}{2}}$$

$$= E\left\{ \log((2\pi)^n |K|)^{1/2} + \frac{1}{2}(\underline{x} - \underline{\mu}^T) K^{-1} (\underline{x} - \underline{\mu}) \log e \right\}$$

$$= \frac{1}{2}\log (2\pi)^n |K| + \frac{1}{2} \log e \; E\left[(\underline{x} - \underline{\mu}^T) K^{-1} (\underline{x} - \underline{\mu})\right]$$

<u>Note:</u> $\quad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$u^t v = \mathrm{Trace}(u v^t)$$

$$\Rightarrow E\left[(\underline{x} - \underline{\mu})^T K^{-1} (\underline{x} - \underline{\mu})\right] = E\left[\mathrm{Tr}\left[(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T (K^{-1})^T\right]\right]$$

$$= \mathrm{Tr}\left[E\left[(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T K^{-1}\right]\right]$$

$$= \mathrm{Tr}\left[K K^{-1}\right] = n$$

$$\Rightarrow h(\underline{x}) = \frac{1}{2} \log (2\pi)^n |K| + \frac{n}{2} \log e = \frac{1}{2} \log (2\pi e)^n |K|$$

Natural