Lec 23: Guassian channel; converse, band limited case, parallel channel case. ①

Recap:

- Bound on estimation error (mean square) in terms of entropy
  - $\left\{\begin{array}{l}\text{conditional entropy when there is side} \\ \text{information}\end{array}\right.$
- Guassian channel

$$X_i \longrightarrow \oplus \xrightarrow{\;Y_i\;} \qquad \uparrow Z_i \sim \mathcal{N}(0, N)$$

- Information Capacity
  - $\frac{1}{2}\log\left(1+\frac{P}{N}\right)$
- Achieving capacity using a random Guassian codebook.

$$\boxed{\text{Source}} \xrightarrow{W} \boxed{\text{Encoder}} \xrightarrow{\underline{X}} \boxed{\text{Channel}} \xrightarrow{\underline{Y}} \boxed{\text{Decoder}} \xrightarrow{\hat{W}}$$

Converse: Once cannot transmit information at reliably at rate R if

$$R > C = \frac{1}{2}\log\left(1+\frac{P}{N}\right) \qquad \text{constraint } E[X^2] \le P.$$

Proof: $nR = H(W) = H(W|\hat{W}) + I(W;\hat{W})$

(aside; Fano's inequality)

$$E = \begin{cases} 1 & W \neq \hat{W} \\ 0 & \text{else} \end{cases}$$

$$H(W, E|\hat{W}) = H(W|\hat{W}) + \underbrace{H(E|W,\hat{W})}_{0}$$

$$= H(E|\hat{W}) + H(W|E,\hat{W})$$

$$\Rightarrow H(W|\hat{W}) \le \underbrace{H(E) + \Pr(E=1)\, H(W|\hat{W}, E=1)}_{\le 1 + P_e \log(M-} \le 1 + P_e nR.$$

$$\therefore H(W|\hat{W}) \le 1 + P_e n R = n\epsilon_n \text{ (say)}$$

Here W takes any of the 2^{nR} possibilities equally likely.

$$\begin{aligned}
\therefore nR &\le I(W;\hat{W}) + n\epsilon_n \\
&\le I(\underline{X};\underline{Y}) + n\epsilon_n \quad \text{[data processing inequality]} \\
&= h(\underline{Y}) - h(\underline{Y}|\underline{X}) + n\epsilon_n \\
&= h(\underline{Y}) - h(\underline{X}+\underline{Z}|\underline{X}) + n\epsilon_n \\
&= h(\underline{Y}) - h(\underline{Z}|\underline{X}) + n\epsilon_n \\
&= h(\underline{Y}) - h(\underline{Z}) + n\epsilon_n. \\
&\le \sum_{i=1}^{n} h(Y_i) - \sum_{i=1}^{n} h(Z_i) + n\epsilon_n.
\end{aligned}$$

<u>Codebook</u>

$\rightarrow$ ith channel use.

$$M = 2^{nR} \quad \begin{bmatrix} x_1(1) & \cdots & \cdot & \cdot & x_n(1) \\ \vdots & & & & \\ x_1(M) & \cdots & \cdot & \cdot & x_n(M) \end{bmatrix}$$

P_i

$\longleftarrow n \longrightarrow$

$$E\left[X_i^2(W)\right] = \frac{1}{M}\sum_{w=1}^{M} x_i^2(w) = P_i \text{ (say)}$$

Have the constraint

$$\frac{1}{M}\sum_{w=1}^{M}\sum_{i=1}^{n} x_i^2(w) \le Pn$$

$$\therefore \sum_{i=1}^{n} P_i \le Pn$$

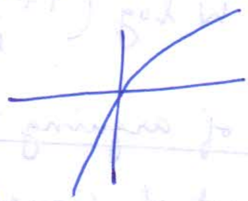
$\therefore\ h(Y_i) \le \frac{1}{2}\log 2\pi e\,(P_i + N)$ as $E(Y_i^2) = E[(X_i+Z_i)^2]$
$= P_i + N.$

$h(Z_i) = \frac{1}{2}\log 2\pi e N.$

$$\therefore\ nR \le \frac{1}{2}\sum_{i=1}^{n}\log 2\pi e(P_i+N) - \frac{n}{2}\log(2\pi e N) + n\epsilon_n$$

$$\Rightarrow\ R \le \frac{1}{2}\sum_{i=1}^{n}\frac{1}{n}\log \cancel{2\pi e}\left(1+\frac{P_i}{N}\right) + \epsilon_n.$$

let $f(x_i) = \log\left(1+\frac{x_i}{N}\right)$ is a concave function



$$\Rightarrow\ \sum_{i=1}^{n}\frac{1}{n}\log\left(1+\frac{x_i}{N}\right) \le \log\left(1+\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{N}\right)$$

$$\Rightarrow\ R \le \frac{1}{2}\cancel{\sum_{i=1}^{n}}\log\cancel{2\pi e}\left(1+\frac{\sum_{i=1}^{n}P_i}{nN}\right) + \epsilon_n.$$ [by concavity]

$$\le \frac{1}{2}\cancel{\Sigma}\log\left(1+\frac{P}{N}\right) + \epsilon_n$$ [as log is increasing function].

$$\therefore\ \epsilon_n \ge R - \frac{1}{2}\log\left(1+\frac{P}{N}\right).$$

$$\frac{1}{n} + P_e R \ge R - \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

P_e is bounded away from zero if $\underline{R > C}$

<u>Capacity of a bandlimited channel.</u>

$$Y(t) = (X(t) + Z(t)) * h(t).$$

time limited

frequency (bandwidth) limited

$n(t) \rightarrow$ white guassian noise psd $= \frac{N_0}{2}$

Gallager has a rigorous proof

End result

Equivalent to $2WT$ Guassian channels

$$2WT \, \frac{1}{2} \log\left(1 + \frac{P}{WN_0}\right)$$

$$E\left[X^2(t)\right] \le P$$

$$C = W \log\left(1 + \frac{P}{WN_0}\right) \text{ bits per sec.}$$

One way of arguing this:

The set of signals that are both time & bandwidth limited in this fashion (a vector space with dimensionality, 2WT) is $\simeq 2WT = D$.

$$y(t) = x(t) + n(t)$$

let $\{\varphi_i(t)\}_{i=1}^{2WT}$ be an ~~orthogonal~~ orthonormal basis for this vector space.

$$\int_0^T \varphi_i(t)\,\varphi_j(t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$x_i = \int_0^T x(t)\,\varphi_i(t)\,dt$$

$$x(t) = \sum_{i=1}^{D} x_i\,\varphi_i(t)$$

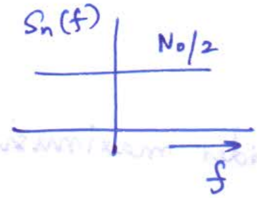
$$E\left[\int_0^T x^2(t)\,dt\right] = \sum_{i,j} \int_0^T E\left[x_i x_j\right] \varphi_i(t)\,\varphi_j(t)\,dt$$

$$= \int_0^T E\left[x^2(t)\right] dt \qquad = E\left[\sum_{i=1}^{D} x_i^2\right] \le PT$$

$$\le PT$$

$$n_i = \int n(t)\,\varphi_i(t)\,dt$$

$$E[n_i n_j] = \int_0^T \int_0^T \underbrace{E[n(t_1)\,n(t_2)]}_{\Downarrow\ \frac{N_0}{2}\delta(t_1 - t_2)} \varphi_i(t_1)\,\varphi_j(t_2)\,dt_1\,dt_2$$



$$= \frac{N_0}{2} \int_0^T \varphi_i(t)\,\varphi_j(t)\,dt$$

$$= \frac{N_0}{2}\,\delta_{i,j}$$

$$\therefore\ x(t) + n(t) \longrightarrow \underbrace{x_i + n_i}_{\Downarrow\ \text{iid } \eta\left(0, \frac{N_0}{2}\right)} \qquad 1 \le i \le D.$$

$$\sum_{i=1}^{D} \underbrace{E[x_i^2]}_{\Downarrow\ E_i} \le PT$$

$$\therefore\ C \le \frac{D}{D} \sum_{i=1}^{D} \frac{1}{2} \log\left(1 + \frac{E_i}{\frac{N_0}{2}}\right)$$

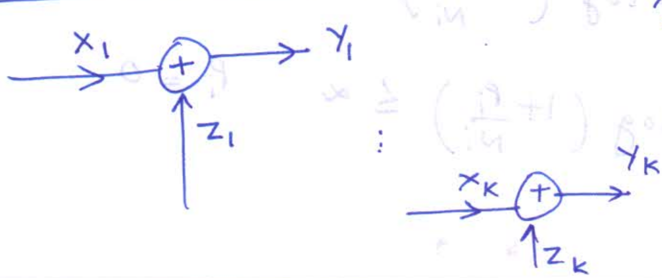
$$\le \frac{D}{2} \log\left(1 + \frac{\sum_{i=1}^{D} E_i}{D\,\frac{N_0}{2}}\right)$$

$$\le \frac{D}{2} \log\left(1 + \frac{PT}{2WT\,\frac{N_0}{2}}\right)$$

$$= WT \log\left(1 + \frac{P}{WN_0}\right).$$

$$\Rightarrow\ C = W \log\left(1 + \frac{P}{WN_0}\right) \text{ bits per sec.}$$

Parallel Guassian channels



$$Y_j = X_j + Z_j \qquad 1 \le j \le k$$

$$Z_j \sim \eta(0, N_j)$$

$$E\left(\sum_{j=1}^{k} X_j^2\right) \le P.$$

$C = \max I(X;Y)$

$f(x)$:

$\sum_{j=1}^{k} E[X_j^2] \le P$

$$I(\underline{X};\underline{Y}) = h(\underline{Y}) - h(\underline{Y}|\underline{X})$$

$$\le \sum_{i=1}^{k} h(Y_i) - \sum_{i=1}^{k} h(Z_i)$$

$$\le \frac{1}{2}\sum_{i=1}^{k}\{\log 2\pi e(P_i+N_i) - \log 2\pi e N_i\}$$

$$= \sum_{i=1}^{k}\frac{1}{2}\log\left(\frac{P_i+N_i}{N_i}\right)$$

Consider maximising

$$\sum_{i=1}^{k} \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right)$$

Subject to $\sum_{i=1}^{k} P_i \le P$.

Clearly ~~without~~ WLOG can assume $\sum_{i=1}^{k} P_i = P$

$\therefore$ we wish to maximize

$$\sum_{i=1}^{k} \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right)$$

Subject to $\sum_{i=1}^{k} P_i = P$, $P_i \geq 0$.

Note that

$\sum_{i=1}^{k}\frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right)$ is a concave fn of the prob. vector $\left(\frac{P_1}{P}, \ldots \frac{P_k}{P}\right)$ Gallegar shows that the sufficient condition of the maximum is to find ρ st

$$\frac{\partial}{\partial P_i}\frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right) = \alpha \quad P_i \neq 0$$

$$\frac{\partial}{\partial P_i}\frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right) \le \alpha \quad P_i = 0$$

$$\frac{\partial}{\partial P_i} \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right)$$ (WLOG assume logs to base e.)

$$= \frac{1}{2}\,\frac{1}{\left(1+\frac{P_i}{N_i}\right)}\,\frac{1}{N_i} = \frac{1}{2(P_i+N_i)}$$

$$\therefore \quad \frac{1}{2(P_i+N_i)} = \alpha \qquad P_i \neq 0$$

$$\frac{1}{2(P_i+N_i)} \le \alpha \qquad P_i \doteq 0$$

let $\frac{1}{2\alpha} = v$.

Then solution is $P_i = (v - N_i)^+$

choose v st.

$\sum_{i=1}^{k} P_i = P$. This is called the waterfilling allocation of power.