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lec 9: Entropy rate

Recap

* stationary random sequence $\{x_n\}$ (random process)
* Markov chain or markov process
* Time invariant markov chains
  - irreducible, aperiodic, example
  - stationary distribution

* entropy rate

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{H(x_1, \dots x_n)}{n}$$ when the limit exists

$= H(X_1)$ if $\{X_n\}$ iid

- saw an example where X_i's are independant but not identical – where limit does not exist

TODAY

- specialize to case of stationary process.
- Show that the limit corresponding to $H(\mathcal{X})$ exists.
- Compute for a stationary Markov chain (Example)

Define:

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n \mid X_1 \dots X_{n-1})$$

(when the limit exists)

Thm: For a stationary random process

$$H(\mathcal{X}) = H'(\mathcal{X})$$

Pf: We will first show that $\lim_{n\to\infty} H(X_n \mid X_1 \dots X_{n-1})$ exists

Note that $0 \le H(X_{n+1} \mid X_1 \dots X_n) \le H(X_{n+1} \mid X_2, \dots X_n)$ (conditioning reduces entropy)

(by stationarity) $= H(X_n \mid X_1, \dots X_{n-1})$. monotonically

$\therefore$ $H(X_n \mid X_1, \dots X_{n-1})$ is a decreasing (non-increasing) sequence. and is lower bounded by 0.

Hence the limit exists

(See fundemental concepts of Analysis, Rudin)

Lemma: (Cesaro mean)

If $a_n \to a$ and $b_n \triangleq \frac{\sum_{i=1}^{n} a_n}{n}$, then $b_n \to a$ as well.

Goal is to show that

$$\lim_{n\to\infty} \frac{H(X_1, \dots X_n)}{n} \longrightarrow H'(\mathcal{X}) \text{ as well}$$

$$\lim_{n\to\infty} \frac{H(X_1, \dots X_n)}{n} = H(\mathcal{X})$$

Pf (of lemma).

<u>Goal</u>: Given any $\epsilon > 0$, to show that $\exists n_\epsilon \in \mathbb{N}$ such that $\forall n \geq n_\epsilon$

$$|b_n - a| \leq \epsilon$$

Consider

$$|b_n - a| = \left| \frac{\sum_{k=1}^{n} a_k}{n} - a \right| = \frac{1}{n} \left| \sum_{k=1}^{n} (a_k - a) \right|$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} |a_k - a|$$

Now choose M_ϵ st $|a_k - a| \leq \epsilon/2 \quad \forall k > M_\epsilon$

then $$\frac{1}{n} \sum_{k=1}^{n} |a_k - a| = \frac{1}{n} \sum_{k=1}^{M_k} |a_k - a| + \frac{1}{n} \sum_{k=M_k+1}^{n} |a_k - a|$$

$$\leq \frac{M_k}{n} \max_{k=1}^{M_k} (a_k - a) + \frac{\epsilon}{2}$$

Now choose n_ϵ large enough s.t

st $$\frac{M_k}{n_\epsilon} \max_{k=1}^{M_k} (a_k - a) \leq \epsilon/2$$

ie, $$n_\epsilon > \frac{2M_k}{\epsilon} \max_{k=1}^{M_k} (a_k - a)$$

for that n_ϵ, $\forall n \geq n_\epsilon$

$$|b_n - a| \leq \epsilon.$$

Assume $a_n = H(X_n | X_1 \ldots X_{n-1})$.

then $b_n = \frac{H(X_1, \ldots X_n)}{n}$, It follows that

$$H(\mathcal{X}_0) = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = H'(\mathcal{X}_0) \quad //$$

Next, we focus on the case of a markov chain <u>that is stationary</u>.

for a markov chain (we assume time invariance)

$$H(X_n | X_1 \ldots X_{n-1}) = H(X_n | X_{n-1})$$

by time invariance

$$= H(X_2 | X_1)$$

$$= \sum_i P_{X_1}(i) \sum_j P_{X_2|X_1}(j|i) \log P_{X_2|X_1}(j|i)$$

$$= \sum_{x_{n-1}} p_{X_{n-1}}(x_{n-1}) \sum_{x_n} p_{X_n|X_{n-1}}(x_n | x_{n-1}) \cancel{\log p(x_0)} \log p_{X_n|X_{n-1}}(x_n | x_{n-1})$$

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~~Hence the limit is easily~~

$$H(X_n \mid X_1 \ldots X_{n-1}) = H(X_n \mid X_{n-1}) \quad \{\text{By definition of markov chain}\}$$

$$= \sum_{x_{n-1}} P_{X_{n-1}}(x_{n-1}) \sum_{x_n} P_{X_n|X_{n-1}}(x_n|x_{n-1}) \log \frac{1}{P_{X_n|X_{n-1}}(x_n|x_{n-1})}$$

by stationarity $\updownarrow$ $\qquad$ $\updownarrow$ by time invariance

$$= \sum_{x_{n-1}} P_1(x_{n-1}) \sum_{x_n} P_{X_2|X_1}(x_n|x_{n-1}) \log \frac{1}{P_{X_2|X_1}(x_n|x_{n-1})}$$

$$= \sum_{x_1} P_1(x_1) \sum_{x_2} P_{X_2|X_1}(x_2|x_1) \log \frac{1}{P_{X_2|X_1}(x_2|x_1)}$$

$= H(X_2|X_1)$ Hence the limit is easily computed

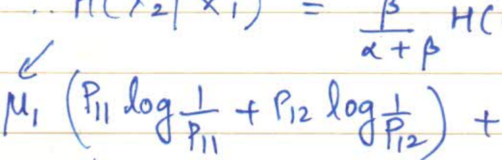
let $[\mu_1, \mu_2, \ldots \mu_m]$ represent the stationary distribution of the markov chain. then

$$\mu_j = \sum_{i \in \mathcal{X}}^{m} \mu_i P_{ij} \qquad m = |\mathcal{X}|$$

$$P_{ij} = P(j|i)$$

$$\therefore H(X_2|X_1) = \sum_i \mu_i \sum_j P_{ij} \log \frac{1}{P_{ij}} = \sum_i \mu_i H(P_{i1}, P_{i2}, \ldots P_{im})$$

Eg:

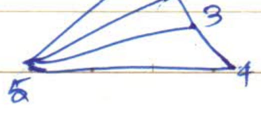


$$\mu_1 = \frac{\beta}{\alpha+\beta}, \quad \mu_2 = \frac{\alpha}{\alpha+\beta}.$$

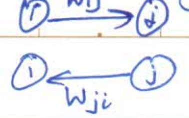
$$\therefore H(X_2|X_1) = \frac{\beta}{\alpha+\beta} H(\alpha) + \frac{\alpha}{\alpha+\beta} H(\beta).$$

$$\mu_1 \Big(\underbrace{P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}}}_{H(\alpha,\, 1-\alpha)}\Big) + \mu_2 \Big(\underbrace{P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}}}_{H(\beta,\, 1-\beta)}\Big).$$

Eg: Consider a random walk on graph.



Assume that the weights are given by W_{ij}



Assume an undirected graph i.e., $w_{ij} = w_{ji}$

Define $w_i = \sum_j w_{ij}$ (the sum of weights of outgoing edges)

Prob $i \to j = P_{ij} = P(j|i) = \dfrac{w_{ij}}{\sum_j w_{ij}}$

What is the stationary distribution?

$$\pi_j = \sum_i \pi_i P_{ij} = \sum_i \pi_i \frac{w_{ij}}{w_i}$$ ~~uniform~~

$$1 = \cancel{\sum_j \pi_j} = \cancel{\sum_j} \sum_i \cancel{\pi_i} \frac{w_{ij}}{w_i} = \cancel{\sum_i \pi_i}$$

$$\pi_j P_{ji} = \pi_i P_{ij}$$
$$\pi_j \frac{\cancel{w_{ji}}}{w_j} = \pi_i \frac{\cancel{w_{ij}}}{w_i}$$
$$\Rightarrow \frac{\pi_j}{\pi_i} = \frac{w_j}{w_i}$$

Define: $\sum_{i>j} w_{i,j} = W \Rightarrow \sum_i w_i = 2W$.

Guesses

$$\pi_i = \frac{w_i}{2W}$$

Need that

$$\pi_j = \sum_i \pi_i P_{ij} = \sum_i \frac{w_i}{2W} \frac{w_{ij}}{w_i} = \frac{\sum_i w_{ij}}{2W} = \frac{w_j}{2W}.$$

Thus the stationary distribution is $\left\{\dfrac{w_i}{2W}\right\}_{i=1}^{m}$.

What is $H(X_2|X_1)$

$$H(X_2|X_1) = \sum_{i=1}^{m} \pi_i \sum_{j=1}^{m} P_{ij} \log \frac{1}{P_{ij}}$$

$$= \sum_{i=1}^{m} \frac{w_i}{2W} \sum_{j=1}^{m} \frac{w_{ij}}{w_i} \log \frac{w_i}{w_{ij}}$$

twice # of edges

$$\underbrace{H\left(\ldots \frac{w_{ij}}{2W} \ldots\right)}$$

$$= \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{w_{ij}}{2W} \log \frac{w_i}{w_{ij}}} = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{w_{ij}}{2W} \log \frac{2W}{w_{ij}} - \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{w_{ij}}{2W} \log \frac{2W}{w_i}}_{H\left(\ldots, \frac{w_i}{2W}, \ldots\right)}$$

$$= H\left(\ldots, \frac{w_{ij}}{2W}, \ldots\right) - H\left(\ldots \frac{w_i}{2W} \ldots\right)$$

m terms
# of nodes.

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Suppose all wts are 1.

$W = \#\text{ edges} = E \qquad 2W = 2E.$

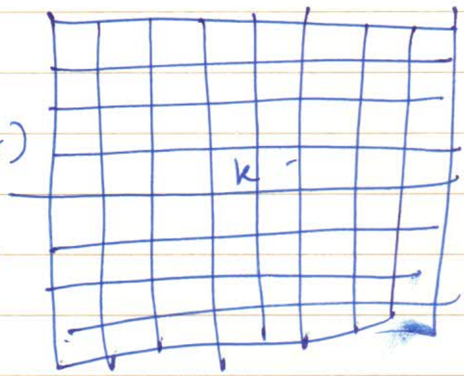
$W_{ij} = 1 \qquad \sum_j W_{ij} = d_i$ (degree of node i).

$H(X_2 | X_1) = \log 2W - H\left(\frac{d_1}{2E}, \ldots \frac{d_m}{2E}\right)$

(corner) 4 → 3 moves

24 → 5 moves

36 (Interior) → 8 moves



chess board

king K moves

$2 \times \#\text{ of edges} = 4 \times 3 + 24 \times 5 + 36 \times 8$

$= 12 + 120 + 288 = 420.$

$\underline{\underline{E = 210.}}$

$\log(\cancel{420}) - H(\ldots)$

$= 0.52 \log 8$