E1–244	**Detection and Estimation Theory**	Discussions: None Due: 10 Feb 2015

Problem Set 2

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Problems:

1. Consider the composite hypothesis testing problem with parameter space Λ and distributions $\{p_\theta : \theta \in \Lambda\}$, a family of distributions on the space $(\Gamma, \mathcal{G})$. Let Λ_0, Λ_1 be a partition of Λ such that
$$H_j : Y \sim p_\theta, \quad \theta \in \Lambda_j, \text{ for } j = 0, 1.$$
Suppose that the prior distribution for the parameter is known and given by the density $w(\theta)$ on Λ. Let $C(j, \theta)$ be cost of deciding H_j when the true parameter is θ. Show that the decision rule with the following critical region minimises the Bayes risk:
$$\Gamma_1 = \{y \in \Gamma \mid \mathbb{E}[C(1, \Theta)|Y = y] < \mathbb{E}[C(0, \Theta)|Y = y]\}.$$

2. Can you show that the zeroth order modified Bessel function of the first kind $I_0(x)$ is a monotone increasing function of its argument for $x \geq 0$?

3. Problem 20 in Section II.F

4. Find the locally most powerful detector for the following problem.
$$\begin{aligned} H_0 \quad &: \quad Y^n = Z^n \\ &\text{vs.} \\ H_1 \quad &: \quad Y^N = \theta s^n + Z^n \end{aligned}$$
where s^n is a known real vector, $\theta > 0$, and $Z^n = (Z_1, Z_2, \ldots, Z_n)$ is iid Cauchy distributed. Interpret your answer.

5. Verify that the inverse of an invertible lower triangular matrix is lower triangular.

6. Problem 1 in Section III.F.

7. Consider the following hypothesis testing problem:
$$H_0 : \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s + Z_1 \\ Z_1 + Z_2 \end{pmatrix} \quad \text{versus} \quad H_1 : \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} -s + Z_1 \\ Z_1 + Z_2 \end{pmatrix}$$
where $s > 0$ is known. Assume that both hypotheses are equally likely and that Z_1 and Z_2 are independent $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$ random variables. Does Y_2 provide any useful information? Justify your answer. (If it is useful, the probability of error for some detector that uses both Y_1 and Y_2 should be strictly smaller than that of the optimal detector based only on Y_1. The point of this question is to investigate if something made of only noise (Y_2) can ever be useful).

8. *Programming problem*: A psychic claimed that he could guess the suit of a card correctly with an unknown probability $q > 1/4$. An experiment was conducted to test his claim. A card was picked at random from a normal deck of 52 cards. The card's suit and the psychic's guess were noted. The card was then replaced and the deck was well-shuffled. The above procedure was repeated for a total of 100 tries. The data is available in the file:
http://www.ece.iisc.ernet.in/˜rajeshs/E1244/psychic_data_ps2.txt

   The first column contains the actual suits of the drawn cards. The second column contains the psychic's guesses. The letters 'C', 'D', 'H', 'S' stand for 'Clubs', 'Diamonds', 'Hearts', 'Spades', respectively.

   (a) Specify your null and alternate hypotheses clearly. Design your test of level $\alpha = 0.2$. What is your decision on the psychic's claim?

   (b) Suppose that you give the psychic the allowance of any permutation on the suits that is consistent through the experiment. (For example, he may consistently use 'C', 'H', 'S', 'D' as the labels for the suits 'C', 'D', 'H', 'S', respectively.) This suggests a change in the test statistic. Clearly specify your test statistic. Also, clearly specify your null and alternate hypotheses. Design a test of level $\alpha = 0.2$ based on the new statistic. What is your new decision on the psychic's claim?