E1–244	**Detection and Estimation Theory**	
		Discussion: Mondays 4:00 - 5:00 PM No due date

Problem Set 4

Instructor: Rajesh Sundaresan — *TA: None*

Problems: With something to think about on each question.

1. Give an approximation for $\mathbb{E}[N|H_0]$ analogous to the approximation $\mathbb{E}[N|H_1]$ obtained in class. Explain why the two expressions are different (both numerator and denominator).

2. Suppose that the target false alarm rate and miss probabilities are identical and given by ε. Assume that the two hypotheses have priors π_0 and $\pi_1 = 1 - \pi_0$. Give an approximation for the limiting value
$$\lim_{\varepsilon \to 0} \frac{\mathbb{E}[N]}{\log(1/\varepsilon)}.$$
This not only tells how the number of samples grows as ε shrinks, but also gives the proportionality constant.

3. Consider the following sequential detection problem.
$$\begin{aligned} H_0 \quad : \quad Y_k &= Z_k, \quad k = 1, 2, \ldots \\ &\text{versus} \\ H_1 \quad : \quad Y_k &= \theta + Z_k, \quad k = 1, 2, \ldots, \quad \theta > 0. \end{aligned}$$
where Z_k are iid $N(0, 1)$. Find $D(P_1||P_0)$ and $D(P_0||P_1)$. Does the answer surprise you? Assuming that the false alarm rate and miss probability are ε give an expression for the (approximate) expected number of samples for a decision.

4. For the hypothesis testing problem above, take $\theta = 1$. But consider a fixed number of samples n_0. Using an expression relating power and size of a fixed sample test, describe how to obtain n_0 so that the false alarm rate and miss probability are both ε. For small ε, say $\varepsilon = 0.01$ compare n_0 of this problem and $\mathbb{E}[N|H_j]$ of the previous problem. Which is better?

5. A proof of the Wald-Wolfowitz theorem (Burkholder and Wijsman 1963).
Step 1: In class we considered the case of uniform costs. Consider the more general case where $C_{01} = w$, $C_{10} = 1 - w$, and $C_{11} = C_{00} = 0$. Let c be the cost per sample. For a fixed w, verify that the optimal sequential decision rule is an SPRT test with $\pi_L(c, w) \leq \pi_U(c, w)$ as the lower and upper thresholds, respectively, on the posterior probability.

6. *Step 2*: Let the prior π_1 satisfy $\pi_L(c, w) \leq \pi_1 \leq \pi_U(c, w)$. Identify the threshold A and B for the likelihood ratio in the SPRT(A, B).

7. *Step 3*: For a fixed w, assume that $\pi_L(c, w)$ and $\pi_U(c, w)$ are continuous, and further assume that $\lim_{c\to 0} \pi_L(c, w) = 0 = 1 - \lim_{c\to 0} \pi_U(c, w)$. Is this reasonable?
Fix any $\varepsilon > 0$ and $0 < A \leq 1 \leq B < +\infty$, show that there exist (a) π_1, c, w having $0 < \pi_1 < \varepsilon$ and A, B as given by the formulae in the previous step.

8. *Step 4*: Consider now an SPRT (ϕ, δ) and any other test (ϕ', δ') such that
$$P_F(\phi', \delta') \leq P_F(\phi, \delta) \text{ and } P_M(\phi', \delta') \leq P_M(\phi, \delta).$$
Using the previous steps to argue that $\mathbb{E}_0[N(\phi')] \geq \mathbb{E}_0[N(\phi)]$. Outline the steps to prove the inequality under H_1.