Problem Set 10

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1. Consider the case considered in the lecture - finite maximum distortion and finite alphabet. Deduce from the properties of $R(D)$ that it is strictly decreasing until it hits zero.

2. Prove that the function

$$S(D) := \min_{P_{Y|X}:E[d(X,Y)]\leq D} I(X;Y)$$

   is nonincreasing and convex in D.

3. *One-bit quantization of a single Gaussian random variable*: Let $X \sim \mathcal{N}(0, \sigma^2)$ and let the distortion measure be squared error. Here we do not allow block descriptions. Show that the optimum reproduction points for 1-bit quantization are $\pm\sqrt{\frac{2}{\pi}}\sigma$ and that the expected distortion for 1-bit quantization is $\frac{\pi-2}{\pi}\sigma^2$. Compare this with the distortion rate bound $D = \sigma^2 2^{-2R}$ for $R = 1$.

4. *Lloyd-Max algorithm*: Let X be a random variable with probability density function f_X. Suppose we wish to quantise X to L levels. The goal is to minimise the mean squared distortion $\mathbb{E}[(X - \hat{X})^2]$. Let $\mu_1^0, \mu_2^0, \cdots, \mu_L^0$ be the L initial representative points. Consider the following iterative procedure. For $k = 1, 2, \ldots$:

   - Identify the subsets $A_1^k, A_2^k, \cdots, A_L^k$ that should map to the representatives $\mu_1^{k-1}, \mu_2^{k-1}, \ldots, \mu_L^{k-1}$, respectively.
   - Identify the new representatives $\mu_1^k, \mu_2^k, \cdots, \mu_L^k$ for the subsets $A_1^k, A_2^k, \cdots, A_L^k$, respectively.

   Solve the two individual steps of the iterative procedure.

5. For $x, y \in [0, 1]$, show that $(1 - xy)^m \leq 1 - y + e^{-mx}$.

6. *Rate distortion function with infinite distortion.*
   Find the function $S(D)$ for $X \sim Ber(\frac{1}{2})$ and distortion

$$d(x, \hat{x}) = \begin{cases} 0, & x = \hat{x} \\ 1, & x = 1, \hat{x} = 0 \\ \infty, & x = 0, \hat{x} = 1. \end{cases}$$

7. *Rate distortion for binary source with asymmetric distortion.*

   Fix $p(\hat{x}|x)$ and evaluate $I(X; \hat{X})$ and D for $X \sim Ber(\frac{1}{2})$, $d(x, \hat{x}) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$

8. *Rate distortion for uniform source with Hamming distortion.*
   Consider a source X uniformly distributed on the set $\{1, 2, \ldots, m\}$. Find the rate distortion function for the source with Hamming distortion, i.e $d(x, \hat{x}) = 1\{x \neq \hat{x}\}$.

9. *Erasure distortion.*
   Consider binary source but with $\hat{X} \in \{1, 0, e\}$ and we use erasure distortion function
   $$d(x, \hat{x}) = \begin{cases} 0, & x = \hat{x} \\ 1, & \hat{x} = e \\ \infty, & \text{otherwise.} \end{cases}$$

10. *Variational inequality.*
    Verify for positive random variables X that $\log E_P(X) = \sup_Q [E_Q(\log X) - D(Q\|P)]$.

    This variational characterisation is of fundamental importance in statistical mechanics.