E2–201 Information Theory Discussion: Saturday 29 March 2025

Problem Set 8

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1. Prove the following for the $A(n, \delta)$ defined in the continuum case.

$$
\begin{aligned}
&a) \quad \lim_{n\to\infty} P\{A(n,\delta)\} = 1.\\
&b) \quad \mathrm{Vol}(A(n,\delta)) \le 2^{nh(X)+n\delta} \text{ for all } n.\\
&c) \quad \mathrm{Vol}(A(n,\delta)) \ge_n (1-\delta)\cdot 2^{nh(X)-n\delta}
\end{aligned}
$$

2. Problem 8.1 of Cover and Thomas (2nd edition).

3. Problem 8.9 of Cover and Thomas (2nd edition).

4. Problem 8.10 of Cover and Thomas (2nd edition).

5. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be refinements of the partitions $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let X and Y be two real-valued random variables with joint distribution $P_{X,Y}$. Show that

$$I([X]_{\mathcal{P}'}; [Y]_{\mathcal{Q}'}) \ge I([X]_{\mathcal{P}}; [Y]_{\mathcal{Q}}).$$

6. Prove the following saddle-point inequality:

$$I(X; X + Z_G) \le I(X_G; X_G + Z_G) \le I(X_G; X_G + Z),$$

where $X_G \sim N(0, P)$ and X is a random variable with mean 0 and variance P, while $Z_G \sim N(0, \sigma^2)$ and Z is a random variable with mean 0 and variance σ^2. (Assume that in any sum, the constituent random variables are independent of each other). While the first inequality is what you used to show the capacity of the AWGN channel, what conclusion can you draw from the second inequality?

7. Let us try to get a saddle-point inequality for the discrete case. Let the input and output alphabets be of finite size. Fix $P_{Y|X}$. Let P_{X^*} maximise $I(X;Y)$ and P_{Y^*} be the corresponding marginal of Y. Which of the inequalities in the following statement is true?

For any P_X, and any Q,

$$D(P_{Y|X} \| P_{Y^*} \mid P_X) \le D(P_{Y|X} \| P_{Y^*} \mid P_{X^*}) \le D(P_{Y|X} \| Q \mid P_{X^*}).$$

8. Consider an ideal gas made of noninteracting (i.e., they do not collide) particles in a horizontal container of finite length. Each particle is of mass m and executes only horizontal motion. Reflections at the two boundaries are elastic so that at these locations the velocity of the particle gets reversed without a change in magnitude. Suppose that a particle's velocity V is distributed according to the density function p satisfying

   - $\mathbb{E}_p[V] = 0$,
   - $\mathbb{E}_p\left[\frac{1}{2}mV^2\right] = \frac{1}{2}m\sigma^2$; (temperature is a measure of average kinetic energy).

   What is the p satisfying the above conditions that maximises the differential entropy?

9. Let P_X be a PMF on A and $f : A \times B \to \mathbb{R}_+$ a function such that

$$\mathbb{E}\left[f(X, y)\right] = 1, \quad \forall y \in B,$$

   where the expectation is with respect to P_X. Suppose that M codewords $x_1, \cdots, x_M$ are picked, independently of each other, each according to PMF P_X. Let the first codeword be transmitted over the channel $P_{Y|X}$, and let y be received. Let $A_i = \{f(x_i, y) > \beta\}$. Show that

$$P\left\{\cup_{i=2}^{M} A_i \mid \text{ Message } 1\right\} \leq \frac{M}{\beta}.$$

10. (a) For a code transmitting reliably at R bits per second on a channel of passband bandwidth W Hz, average power constraint $\bar{P}$ Watts, what is the energy per bit?
    (b) What is the least energy per bit that is needed for reliable transmission?
    (c) What is bandwidth W needed for a fixed $\bar{P}$ to attain this minimum energy per bit?