E2–202 Random Processes Due: 19 September 2026

Problem Set 5

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Remarks:

- Collaboration, discussion, and working in teams to solve problems is strongly encouraged.
- To test your understanding, practice the solution to each problem in your own words without referring to a friend, text, class notes, or AI engines.

Problems:

1. Given an example of a random variable whose $\mathbb{E}[X]$ exists and is finite but $\mathbb{E}[X^2]$ is infinite.

2. Approximation of random variables.

   X is *simple* if there is a finite $m \geq 1$, events $A_1, \ldots, A_m$ that partition Ω, and real values $x_1, \ldots, x_m$, such that $X(\omega) = \sum_{i=1}^{m} x_i I_{A_i}(\omega)$, for every $\omega \in \Omega$. Further, a simple random variable X is in *canonical form* if the x_i are distinct.

   (a) Suppose X and Y are two simple random variables. Let $a, b \in \mathbb{R}$. Argue that $aX + bY$ is also simple. Describe its canonical form.

   (b) Define for each $n \geq 1$ and $r \geq 0$.

   $$d_n(r) := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(r) + n I_{[n,\infty)}(r), \quad r \geq 0.$$

   Draw $d_n(r)$ and compare it with the function $f(r) = r$ for $r \geq 0$.

   (c) For the above sequence of functions $d_n : [0, \infty) \to [0, \infty)$, show that $d_n \leq d_{n+1}$ for every $n \geq 1$. (I.e., for each $r \geq 0$, we have $d_n(r) \leq d_{n+1}(r)$.)
   Show also that for each fixed $r \geq 0$, we have $d_n(r) \uparrow r$.

   (d) Suppose that

   $$f_n(r) := \sum_{k=1}^{n} \frac{k-1}{n} I_{\left[\frac{k-1}{n}, \frac{k}{n}\right)}(r) + n I_{[n,\infty)}(r), \quad r \geq 0.$$

   Draw f_2 and f_3 and argue that the functional inequality $f_2 \leq f_3$ is not true. Do you see why powers of 2 helped in (c)?

(e) Suppose $X \geq 0$. Show that X is a random variable if and only if there exists a sequence of simple random variables X_n such that $X_n \uparrow X$. (Hint: For one direction, you have a limit of a sequence of random variables. For the other direction, use $X_n = d_n(X) = d_n \circ X$.)

(f) $\mathbb{E}[X]$ is defined as the limit $\{\mathbb{E}[d_n(X)], n = 1, 2, \ldots\}$. Verify that this is consistent with the monotone convergence theorem.

3. Problem 1.9 (in Section 1.8 of the reference text).

4. Let $\mathbb{E}[Y]$ exist and be finite. If $X_n \geq Y$ for all n and $X_n \uparrow X$, then $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Show this.

5. Consider the $\mathbb{Z}$-valued random variable Z with probability mass function

$$P(Z = k) = \frac{C}{1 + |k|^2}, \quad k \in \mathbb{Z},$$

where C is a suitable normalisation constant. Consider $Z_n = Z \cdot I\{|Z| \leq n\}$, where $n \geq 1$.

(a) What is $\mathbb{E}[Z_n]$?
(b) Does $\lim_{n \to \infty} E[Z_n]$ exist? If yes, what is it? If not, why not?
(c) Does $\mathbb{E}[Z]$ exist? If yes, what is it? If not, why not?
(d) What is $\mathbb{E}[|Z|]$?
(e) Explain why $\lim_{n \to \infty} E[|Z_n|] = E[|Z|]$.

6. Suppose that Y_n converges to 0 in probability. Show that

$$\int_0^1 P(|Y_n| > t)\, dt \to 0.$$

Hint: Define $X_n(t) = P(|Y_n| > t)$ and view t as the realisation of a new $U[0, 1]$ random variable.

7. Problem 1.10 (in Section 1.8 of the reference text).

8. Problem 1.11 (in Section 1.8 of the reference text).

9. Problem 1.12 (in Section 1.8 of the reference text).

10. Examine the in-probability and almost-sure modes of convergence for the following random sequences on a common probability space.

(a) Let X_n be a sequence of *independent* binary random variables defined by $P(X_n = 1) = 1/n$ and $P(X_n = 0) = 1 - 1/n$.

(b) Let Y_n be a sequence of not necessarily independent binary random variables given by: $P(Y_n = 1) = 1/n^2$ and $P(Y_n = 0) = 1 - 1/n^2$.