E2–202 Random Processes Due: 03 October 2026

Problem Set 6

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Remarks:

- Collaboration, discussion, and working in teams to solve problems is strongly encouraged.
- To test your understanding, practice the solution to each problem in your own words without referring to a friend, text, class notes, or AI engines.

Problems:

1. Suppose that a fair coin is tossed 10 lakh times (10 lakh = 10, 00, 000). The fair coin has probability that it will land 'Heads' and 'Tails' to be 1/2 and 1/2, respectively. Use the central limit theorem to identify, approximately, the probability that the number of 'Heads' will exceed 500500. Give your answer in terms of the Normal distribution's cdf Φ.

2. One generalisation of the Central Limit Theorem for independent but non-identical random variables is the following.
   Theorem: Let $X_1, X_2, \ldots$ be independent variables satisfying $\mathbb{E}[X_k] = 0$, $\text{var}(X_k) = \sigma_k^2$, $\mathbb{E}[|X_k|^3] < \infty$, and
   $$\lim_{n\to\infty} \frac{1}{\sigma(n)^3} \sum_{k=1}^{n} \mathbb{E}[|X_k|^3] = 0$$
   where $\sigma(n) = \sqrt{\mathsf{Var}(\sum_{k=1}^{n} X_k)}$. Then $\frac{1}{\sigma(n)} \sum_{k=1}^{n} X_k \xrightarrow{d} N(0,1)$ as $n \to \infty$.

   Use this theorem to show the following result: Let $Y_1, Y_2, \ldots$ be independent random variables each taking values $+1$ or -1 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$. Then
   $$\sqrt{\frac{3}{n^3}} \sum_{k=1}^{n} kY_k \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty.$$

3. Exercise 2.1 of Chapter 2 of the reference text.

4. Suppose (X_1, X_2) are jointly Gaussian with mean 0, variances 1 for both components, and correlation ρ. Compute the conditional density of X_1 given "$X_2 = x_2$" and interpret. Note that the conditioning is based on an event of probability 0. How are we approaching this zero-probability event?

5. Let X_1 and X_2 be iid Exp(λ). Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$. Show that the joint density of (Y_1, Y_2) is:

$$f_{Y_1,Y_2}(y_1, y_2) = \lambda^2 y_1 e^{-\lambda y_1} \cdot \frac{1}{(1+y_2)^2}, \qquad y_1 > 0, y_2 > 0.$$

Deduce the marginal densities of Y_1 and Y_2, and further deduce that Y_1 and Y_2 are independent. Deduce $f_{Y_1|Y_2}(y_1|y_2)$.

6. Let X_1 and X_2 be iid Exp(λ). Let $Y_1 = X_1 + X_2$, but let $Y_3 = X_1 - X_2$. Show that the joint density of (Y_1, Y_3) is:

$$f_{Y_1,Y_3}(y_1, y_3) = \frac{1}{2}\lambda^2 e^{-\lambda y_1}, \qquad y_1 > |y_3|.$$

Deduce the marginal densities of Y_1 and Y_3. Deduce $f_{Y_1|Y_3}(y_1|y_3)$.

7. Consider the previous two exercises. Now compute $f_{Y_1|Y_2}(y_1|y_2 = 1)$ and $f_{Y_1|Y_3}(y_1|y_3 = 0)$. Both could be interpreted as the density of X_1+X_2 given "$\{X_1 = X_2\}$". Explain why they are different.

8. Problem 2.1 of Section 2.11 of the reference text.

9. Problem 2.3 of Section 2.11 of the reference text.

10. Problem 2.5 of Section 2.11 of the reference text.