E2 203 Wireless Communication
January-April 2011
Problem Set 4
Due: Tuesday 12 April 2011 at 1:00 PM

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1. Show the following information theoretic inequalities. We used them in the lectures.
   - $H(X, Y) = H(X) + H(Y|X)$
   - $H(X, Y) \leq H(X) + H(Y)$
   - If $P_{Y_1,Y_2|X_1,X_2}(y_1, y_2|x_1, x_2) = P_{Y_1|X_1}(y_1|x_1) \times P_{Y_2|X_2}(y_2|x_2)$ for every y_1, y_2, x_2, x_2 in the respective alphabets, then $H(Y_1, Y_2|X_1, X_2) = H(Y_1|X_1) + H(Y_2|X_2)$.
2. Let X be a complex random vector with covariance K. For simplicity, assume that it has a density and a finite differential entropy. Show that $h(X) \leq \log \det(\pi e K)$ with equality if and only if $X \sim \mathbb{CN}(K)$.
3. Consider the vector Gaussian channel $Y = HX + W \in \mathbb{C}^d$ where $W \sim \mathbb{CN}(I_d)$ and $H = \text{diag}\{h_1, h_2, \ldots, h_d\}$. The power constraint is that the net power across all d dimensions (averaged across time) cannot exceed P. Deduce the water-filling solution for the capacity of the channel.
4. Consider a multipath channel with a finite number of taps, say L. Vectorise the channel to N channel uses, append a cyclic prefix and use OFDM, to get a parallel channel system. Apply the result from the previous question to get an achievable rate. Then let $N \to \infty$ to get an integral expression for the achievable rate. Deduce that this water-filling-in-frequency solution is an achievable rate. [**Bonus**:] Why might it be the capacity of such a channel?
5. Let $Y = HX + W$ where Y is a scalar and $X \in \mathbb{C}^L$. This is the MISO channel. The total power constraint (across all antennas) is P. Evaluate the capacity and interpret the answer.
6. This exercise is in preparation for the outage formulation we will soon see. Suppose that there is one transmitter and two receivers. The channels are $P_{Y_1|X}$ and $P_{Y_2|X}$ respectively. For a given P_X let the respective mutual informations be $I(X; Y_1)$ and $I(X; Y_2)$, respectively. The transmitted message has to be received by both receivers, with high probability. Formulate the definition of capacity for such a requirement. Then adapt the arguments used in class to show $I(X, Y)$ is achievable to this problem, and show that $\min\{I(X; Y_1), I(X; Y_1)\}$ is an achievable rate. [**Bonus**:] Can you guess the capacity for such a system and prove it?