E2–301 Topics in Multiuser Communication	August 30, 2007
	Due: September 06, 2007 4:00 PM

Homework 1

Instructor: Rajesh Sundaresan — *Scribe: Premkumar K.*

1. (**FDMA vs $\mathscr{C}_{\mathbf{MAC}}$**)

   a) Consider a single user communication system having a passband of $\left[f_c - B/2,\ f_c + B/2\right]$Hz and having an average power constraint of $\overline{P}$ Joules/sec. What is the capacity $C(\overline{P}, B)$?

   b) Consider the same system as above, but the passband is $[f_c - B/2, f_c]$Hz. What is the capacity, $C(\overline{P}, B/2)$?

   c) Show that $C(\overline{P}, B/2) \leqslant C(\overline{P}, B)$ by proving $C(\overline{P}, B)$ is an increasing function of B.

   d) Consider a two user FDMA system having an average power constraint $(\overline{P}, \overline{P})$. User 1 uses $[f_c - B/2,\ f_c]$ and user 2 uses $[f_c,\ f_c + B/2]$. Show that $C(\overline{P}, B/2) + C(\overline{P}, B/2) = C(2\overline{P}, B)$. Argue that FDMA achieves the sum capacity and the symmetric capacity (define symmetric capacity as $\sup\{R : (R, R) \in \mathscr{C}_{\mathrm{MAC}}\}$).

   e) Is there any other point on the dominant facet ($R_1 + R_2 = C(2\overline{P}, B)$) that is attained by FDMA.

   (Hint: A passband B requires B complex dimensions per second or $2B$ real dimensions per second)

2. (**Frequency typicality**) For every $\delta > 0$, the following hold for all sufficiently large n. Prove them (Notation is as given in lecture notes).

   (a) $\Pr\left\{Z_{[m]}^n \in T_\delta^{(n)}\right\} \geq 1 - \delta$ and therefore $\Pr\left\{Z_A^n \in T_\delta^{(n)}(Z_A)\right\} \geq 1 - \delta$.

   (b) $z_A^n \in T_\delta^{(n)}(Z_A) \implies \left|\frac{1}{n}\log p_{Z_A^n}(z_A^n) + H(Z_A)\right| \leq \delta$

   (c) $(z_A^n, z_B^n) \in T_\delta^{(n)}(Z_{A\cup B}),\quad A \cap B = \emptyset \implies \left|\frac{1}{n}\log p_{Z_A^n|Z_B^n}(z_A^n|z_B^n) + H(Z_A|Z_B)\right| \leq 2\delta$

   (d) $(1-\delta)2^{nH(Z_A)-n\delta} \leq \left|T_\delta^{(n)}(Z_A)\right| \leq 2^{nH(Z_A)+n\delta}$ so that $\left|T_\delta^{(n)}(Z_A)\right| \doteq 2^{nH(Z_A)\pm 2n\delta}$

   (e) $\widetilde{Z}_{[m]} \sim p_{Z_A} p_{Z_B|Z_A} p_{Z_C|Z_A}$, $A \cup B \cup C = [m], A \cap B = B \cap C = C \cap A = \emptyset, \widetilde{Z}_{[m]}^n$ i.i.d. copies with generic distribution that of $\widetilde{Z}_{[m]}$. Show that $\Pr\left\{\widetilde{Z}_{[m]}^n \in T_\delta^{(n)}\right\} \doteq 2^{-nI(Z_B;Z_C|Z_A)\pm 7n\delta}$

3. (**Conditional frequency typicality**) For every $\delta > 0$, the following hold for all sufficiently large n. Prove them.

   (a) $z_A^n \in T_\delta^{(n)}(Z_A) \implies \Pr\left\{Z_{A^c}^n \in T_{2\delta}^{(n)}(Z_{A^c}|z_A^n) \middle| Z_A^n = z_A^n\right\} \geq 1 - \delta$, so that for any $B \subseteq A^c$, $\Pr\left\{Z_B^n \in T_{2\delta}^{(n)}(Z_B|z_A^n) \middle| Z_A^n = z_A^n\right\} \geq 1 - \delta$.

   (b) $z_A^n \in T_\delta^{(n)}(Z_A)$ and $B \subseteq A^c$, $\implies (1-\delta)2^{nH(Z_B|Z_A)-2n\delta} \leq \left|T_{2\delta}^{(n)}(Z_B|z_A^n)\right| \leq 2^{nH(Z_B|Z_A)+2n\delta}$.

4. Consider (A^n, B^n). Given A_i, the random variable B_i is independent of all other variables, for each $i = 1, 2, \cdots, n$. Prove that

$$I(A^n; B^n) \leq \sum_{i=1}^n I(A_i; B_i)$$

with equality if and only if $B_1, B_2, \cdots, B_n$ are independent.

5. Problem 15.6 (page 598) of Cover and Thomas (2nd edition).