E2–301 Topics in Multiuser Communication August 16, 2007

Lecture 1 : Multiple Access Channels (MAC)

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We begin with some examples.

1 Examples

Example 1 [GMAC] Consider a Gaussian multiple–access channel (GMAC):

$$Y = \sum_{k=1}^{m} X_k + Z. \tag{1}$$

- Each user has an average power constraint P. The average is over codewords and time. $Z \sim N(0, \sigma^2)$
- Users transmit independent data. So the power of $\sum_{k=1}^{m} X_k$ is at most mP. Under this constraint, even if they cooperate, $\sum_{k \in S} R_k \leqslant C\left(\frac{|S|P}{\sigma^2}\right)$, $\forall S \subseteq \{1, 2, 3, \cdots, m\}$, where $C(x) = \frac{1}{2}\log(1 + x)$ is the Shannon capacity at SNR x.
- For $m = 2$: Let $P > 0$.

$$\begin{aligned} R_k &\leqslant \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right), \quad k = 1, 2 \\ R_1 + R_2 &\leqslant \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) + \frac{1}{2}\log\left(1 + \frac{P}{P + \sigma^2}\right) \\ &< 2 \cdot \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) \end{aligned}$$

So $B'B$ will not allow both users to transmit at their respective capacities.

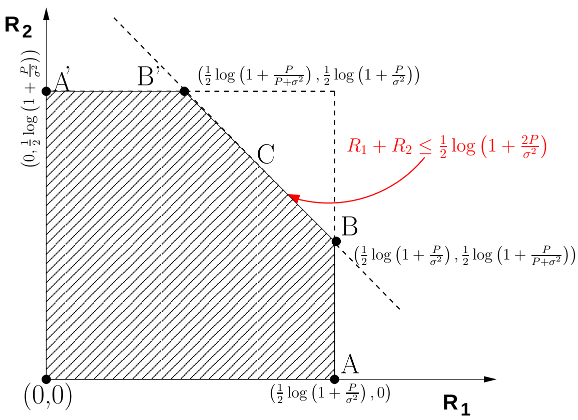

Figure 1: Capacity region (as we will see soon) of two user *Gaussian MAC* with transmit power constraint P and Gaussian channel noise power σ^2.

- Naive TDMA strategy : User 1 uses channel for α fraction of the time. User 2 for $1-\alpha$. Power constraint per (transmitted) sample remains P. The achievable rate is $\big(\alpha C(\frac{P}{\sigma^2}), (1-\alpha)C(\frac{P}{\sigma^2})\big)$.

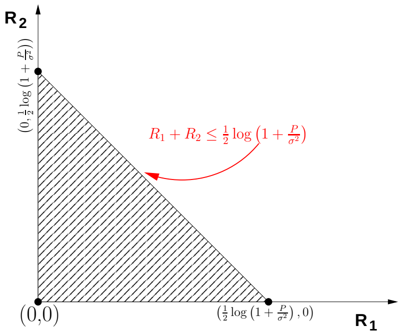

Figure 2: Naive TDMA: Achievable rate region of two user *Gaussian multiple access channel* with transmit power constraint P and Gaussian channel noise power σ^2.

- Smart TDMA: User 1 can transmit at a higher power per sample given that it remains silent for a fraction of time. The average power constraints for both users, when user 1 gets the channel for a fraction α of time is

$$\left(\alpha\left(\frac{P}{\alpha}, 0\right), (1-\alpha)\left(0, \frac{P}{1-\alpha}\right)\right)$$

so that $\left\{(R_1, R_2) : R_1 = \alpha\frac{1}{2}\log\left(1+\frac{P}{\alpha\sigma^2}\right), R_2 = (1-\alpha)\frac{1}{2}\log\left(1+\frac{P}{(1-\alpha)\sigma^2}\right)\right\}$ is achievable.

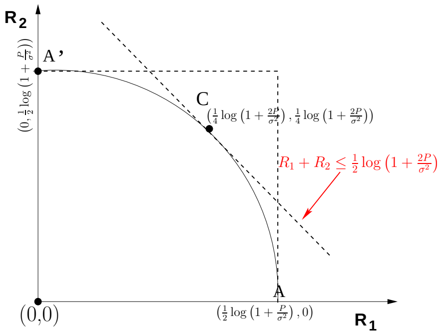

Figure 3: Smart TDMA: Achievable rate region of two user *Gaussian multiple access channel* with transmit power constraint P and Gaussian channel noise power σ^2.

- Note: A, A' are achieved.
- In general,

$$\begin{aligned} R_1 + R_2 &= \alpha\frac{1}{2}\log\left(1+\frac{P}{\alpha\sigma^2}\right) + (1-\alpha)\frac{1}{2}\log\left(1+\frac{P}{(1-\alpha)\sigma^2}\right) \\ &\leqslant \frac{1}{2}\log\left(1+\frac{2P}{\sigma^2}\right), \quad \text{by Jensen's inequality.} \end{aligned}$$

- By the condition for equality in Jensen's inequality, observe that the point C given by $(\frac{1}{4}\log(1+\frac{2P}{\sigma^2}), \frac{1}{4}\log(1+\frac{2P}{\sigma^2}))$ is achieved. C is a point on the outer bound's boundary.

Example 2 [Multiplication Channel]

- $Y = X_1 X_2, \quad X_k \in \{0, 1\}, \; k = 1, 2.$
- In this example, there is no noise; multiple access interference (MAI) is the only source of information corruption.
- The extreme point $A = (1, 0)$ (or $A' = (0, 1)$) can be achieved if user 2 (or user 1) transmits all 1s.
- By time–sharing, any point on the line AA' can be achieved.
- Since Y provides at most one bit of information, we expect $R_1 + R_2 \leqslant 1$. So the triangle OAA' is indeed the capacity region.

Example 3 [Addition Channel]

- $Y = X_1 + X_2, \quad X_k \in \{0, 1\}, \; k = 1, 2$, where the addition is integer addition.
- The extreme point A (or A') can be achieved if user 2 (or user 1) transmits a deterministic sequence.
- No ambiguity if $X_1 = X_2 = 0$ or $X_1 = X_2 = 1$.
- Suppose user 1 sends 1 bit, X_1. User 2's channel is then viewed as an *erasure* channel as shown in Figure 4. An erasure occurs to user 2 whenever both users send different bits. User 2 can thus send at most 1/2 bit. Receiver decodes user 2 first and then user 1 (point B in Figure 5). Similarly point B' can be achieved, and time–sharing gets us the line BB'. From later results, this is indeed the capacity region.

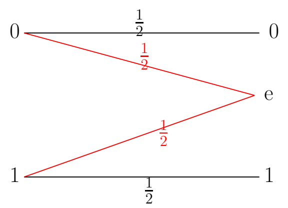

Figure 4: User 2 sees an *erasure channel.*

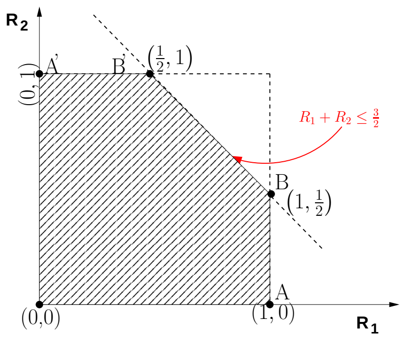

Figure 5: Capacity region of 2 user addition channel.

2 Definitions

Definition 1 (DM–MAC) *A (two user) discrete memoryless multiple access channel (DM–MAC) denoted by* $\left(\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}, p_{Y|X_1X_2}(y|x_1x_2)\right)$*, consists of three finite sets* $\mathbb{X}_1, \mathbb{X}_2$*, and* $\mathbb{Y}$ *and a collection of probability mass functions* $p_{Y|X_1X_2}(\cdot|x_1x_2)$ *on* $\mathbb{Y}$*, one for each* $x_1x_2 \in \mathbb{X}_1 \times \mathbb{X}_2$*, with the interpretation that* X_k *is the input of user* k*,* $k = 1, 2$ *and* Y *is the output. For* $n \in \mathbb{N}$*, with* $X_k^n = (X_{k1}, X_{k2}, \cdots, X_{kn})$*,* $k = 1, 2$ *as inputs, the output sequence* Y^n *has pmf*

$$p_{Y^n|X_1^nX_2^n}(y^n|x_1^nx_2^n) \quad = \quad \prod_{i=1}^{n} p_{Y|X_1X_2}(y_i|x_{1i}x_{2i}) \tag{2}$$

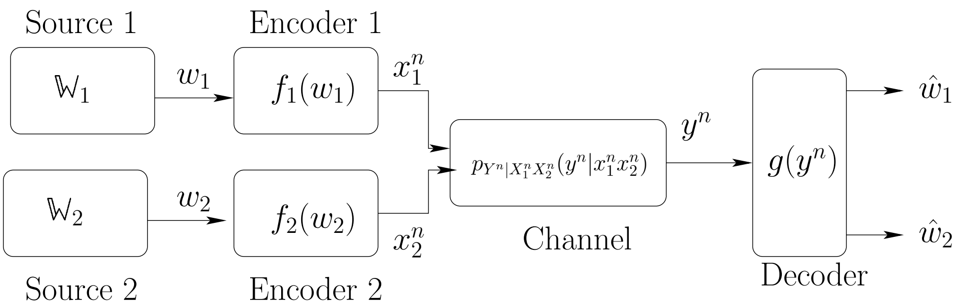

Figure 6: A MAC system diagram.

Definition 2 (Code) *An* (n, M_1, M_2) *code for the channel* $\left(\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}, p_{Y|X_1X_2}(y|x_1x_2)\right)$ *consists of the following:*

1. *An index set of messages for each user* k, $\mathbb{W}_k = \{1, 2, \cdots, M_k\}$.
2. *An encoder* f_k *for each user* k, $f_k : \mathbb{W}_k \to \mathbb{X}_k^n$, $k = 1, 2$. *Note that* $\mathbb{W}_k \ni W_k \mapsto f_k(W_k) \in \mathbb{X}_k^n$. *The codebook can be represented by an ordered set*

$$c = \{f_1(1), f_1(2), \cdots, f_1(M_1); f_2(1), f_2(2), \cdots, f_2(M_2)\}.$$

3. *A decoding rule,* $g : \mathbb{Y}^n \to \phi \cup \left(\mathbb{W}_1 \times \mathbb{W}_2\right)$, *i.e.,* $y^n \mapsto g(y^n) = (\hat{w}_1, \hat{w}_2) \in \phi \cup \left(\mathbb{W}_1 \times \mathbb{W}_2\right)$. *Note that* g *partitions* $\mathbb{Y}^n$ *into decision regions.*

Definition 3 (Probability of error) *Let* W_k *be the message transmitted by user* k *and let* Y^n *be the signal received. The conditional probability of error when* $(W_1W_2) = (w_1w_2)$ *was transmitted is given by*

$$P_{e,w_1w_2}^{(n)}(c) = Pr\{g(Y^n) \neq W_1W_2 | W_1W_2 = w_1w_2\}.$$

The average probability of error for the code c *is given by*

$$P_e^{(n)}(c) = \frac{1}{M_1M_2}\sum_{w_1w_2} P_{e,w_1w_2}^{(n)}(c)$$

Note that the above equation assumes that all messages are equally likely and the users choose their messages independently.

Definition 4 (Achievability) *The rate pair* (R_1, R_2) *is achievable, if for every* $\eta > 0, \lambda \in (0, 1)$, *there exists a sequence of* (n, M_1, M_2) *codes that satisfy*

1. $P_e^{(n)} \leq \lambda$, *and*
2. $\frac{\log_2 M_k}{n} > R_k - \eta$

for all sufficiently large n.

Definition 5 (Capacity region) *The capacity region is the set of all achievable rate pairs, denoted by* $\mathscr{C}_{MAC}$.

3 What can we expect?

- $R_1 + R_2 \leqslant \max_{p(x_1,x_2)} I(X_1X_2; Y)$, (full cooperation)
- $R_k \leqslant \max_{p(x_k)} \max_{x_{k^c}} I(X_k; Y | X_{k^c} = x_{k^c}), \quad k = 1, 2$ (the other user is benevolent)

4 Time Sharing

Lemma 6 *$\mathscr{C}_{MAC}$ is a closed convex set.*

Proof: (convexity) The idea is time sharing. Let $R = (R_1, R_2) \in \mathscr{C}_{\text{MAC}}$ and $R' = (R'_1, R'_2) \in \mathscr{C}_{\text{MAC}}$. Fix $t \in (0,1)$. We will show $tR + (1-t)R' \in \mathscr{C}_{\text{MAC}}$. For a given $(\eta/2, \lambda/2)$, pick a sequence of (n, M_1, M_2) codes and another sequence of (n, M'_1, M'_2) codes such that for all sufficiently large n,

$$\begin{aligned} P_e^{(n)}(c) &\leq \frac{\lambda}{2}, \\ \frac{\log M_k}{n} &> R_k - \frac{\eta}{2}, \\ P_e^{(n)}(c') &\leq \frac{\lambda}{2}, \\ \frac{\log M'_k}{n} &> R'_k - \frac{\eta}{2}. \end{aligned}$$

For each n, use the code of length $\lfloor tn \rfloor$ from the first sequence and the code of length $n - \lfloor tn \rfloor$ from the second sequence. The overall probability of error, $P_e^{(n)}$ is upper bounded by the sum of the individual codes' errors. Since both $\lfloor tn \rfloor$ and $n - \lfloor tn \rfloor \to \infty$, we have for all sufficiently large n,

$$P_e^{(n)} \leq \frac{\lambda}{2} + \frac{\lambda}{2}.$$

Since

$$\begin{aligned} \log M_1 &> \lfloor tn \rfloor (R_1 - \eta/2) \\ \log M_2 &> \lfloor tn \rfloor (R_2 - \eta/2) \\ \log M'_1 &> (n - \lfloor tn \rfloor)(R'_1 - \eta/2) \\ \log M'_2 &> (n - \lfloor tn \rfloor)(R'_2 - \eta/2). \end{aligned}$$

For all sufficiently large n, the overall rate satisfies

$$\begin{aligned} \frac{\log M_k M'_k}{n} &= \frac{\log M_k}{n} + \frac{\log M'_k}{n} \\ &= \frac{\lfloor tn \rfloor}{n} \frac{\log M_k}{\lfloor tn \rfloor} + \frac{n - \lfloor tn \rfloor}{n} \frac{\log M'_k}{n - \lfloor tn \rfloor} \\ &\underset{n}{\geqslant} \frac{\lfloor tn \rfloor}{n} (R_k - \eta/2) + \frac{n - \lfloor tn \rfloor}{n} (R'_k - \eta/2) \\ &\rightarrow t(R_k - \eta/2) + (1-t)(R'_k - \eta/2) \\ &\underset{n}{>} tR_k + (1-t)R'_k - \eta, \quad k = 1, 2 \end{aligned}$$

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