E2–301 Topics in Multiuser Communication	August 22, 2007

Lecture 2 : Multiple Access Channels

Instructor: Rajesh Sundaresan *Scribe: Premkumar K.*

A generalization of the time–sharing notion is the use of a time–sharing random variable, Q that takes values from an arbitrary finite set $\mathbb{Q}$.

Definition 1 ($\mathcal{P}^*$) *$\mathcal{P}^*$:= set of all $Z = QX_1X_2Y$ such that*

(a) X_1 and X_2 are conditionally independent given Q, and

(b) $Pr\{Y = y|Q = q, X_1 = x_1, X_2 = x_2\} = p(y|x_1x_2)$ (conforms to the given channel)

Remark:

$$\begin{aligned}
& \Pr\{Q = q, X_1 = x_1, X_2 = x_2, Y = y\} \\
= \; & \Pr\{Q = q\}\Pr\{X_1 = x_1, X_2 = x_2|Q = q\} \; \Pr\{Y = y|Q = q, X_1 = x_1, X_2 = x_2\} \\
= \; & \Pr\{Q = q\} \underbrace{\Pr\{X_1 = x_1|Q = q\}\Pr\{X_2 = x_2|Q = q\}}_{\text{from } (a)} \underbrace{\Pr\{Y = y|X_1 = x_1, X_2 = x_2\}}_{\text{from } (b)} \\
= \; & p_Q(q)\, p_{X_1|Q}(x_1|q)\, p_{X_2|Q}(x_2|q)\, p_{Y|X_1X_2}(y|x_1, x_2)
\end{aligned}$$

Definition 2 ($\mathscr{C}$**, Capacity region defined by** $\mathcal{P}^*$)

$$\mathscr{C}(Z) := \left\{ \begin{array}{ll} (R_1, R_2): & 0 \leqslant R_1 \leqslant I(X_1; Y|X_2Q), \\ & 0 \leqslant R_2 \leqslant I(X_2; Y|X_1Q), \\ & R_1 + R_2 \leqslant I(X_1X_2; Y|Q) \end{array} \right\}, \quad \textit{a pentagon}$$

$$\mathscr{C} := \textit{closure}\left(\bigcup_{Z \in \mathcal{P}^*} \mathscr{C}(Z) \right)$$

Remarks: In $\mathscr{C}(Z)$, Q serves to average the bounds.

Proposition 3 *Any element of $\mathscr{C}$ is achievable, i.e., $\mathscr{C} \subseteq \mathscr{C}_{MAC}$*

Proof Fix arbitrary $Z = QX_1X_2Y \in \mathcal{P}^*$. It is sufficient to show that $\mathscr{C}(Z)$ is achievable. Fix $R = (R_1, R_2) \in \mathscr{C}(Z)$. Fix $\eta > 0, \lambda \in (0, 1)$. Set $M_k = \lceil 2^{n(R_k - \eta)} \rceil$ so that $\frac{\log M_k}{n} \geq R_k - \eta, k = 1, 2$, for all sufficiently large n. Observe that

$$M_k - 1 \leqslant 2^{n(R_k - \eta)}, \quad k = 1, 2. \tag{1}$$

1. **Codebook generation:**

   - Pick $q^n = (q_1, q_2, \cdots, q_n) \sim \prod_{i=1}^{n} p_Q(q_i)$
   - Pick $x_k^n(w_k) \sim \prod_{i=1}^{n} p_{X_k|Q}(x_{ki}|q_i)$

     Do these pickings independently for $w_k = 1, 2, \cdots, M_k$, $k = 1, 2$. We thus generate the codebook, $c = \{x_1^n(1), x_1^n(2), \cdots, x_1^n(M_1); x_2^n(1), x_2^n(2), \cdots, x_2^n(M_2)\}$. Let C be the code book random variable and let its pmf be p_C.
   - Reveal q^n, c to both encoders and to the decoder.

2. **Encoding:** W_k has the uniform distribution over $\{1, 2, \cdots, M_k\}$, $k = 1, 2$. W_1 and W_2 are independent. Set $f_k(w_k) := x_k^n(w_k)$.

3. **Receiver:** Observe y^n. Let $T_\delta^{(n)}(Z)$ be the frequency typical set (to be defined soon).

   a) Search for a unique $\widehat{w}_1\widehat{w}_2$ such that $q^n x_1^n(\widehat{w}_1)x_2^n(\widehat{w}_2)y^n \in T_\delta^{(n)}(QX_1X_2Y)$. If such a pair exists, set $g(y^n) = \widehat{w}_1\widehat{w}_2$.

   b) If no such $\widehat{w}_1\widehat{w}_2$ exists, set $g(y^n) = \phi$, and declare an error.

4. **Probability of error analysis:**

   a) By symmetry, we can condition on the event that the transmitted messages are 11.

$$\begin{aligned}\mathbb{E}p_e^{(n)}(C) &= \sum_c p_C(c) \sum_{w_1w_2} p_{e,w_1w_2}^{(n)}(c)\frac{1}{M_1M_2} \\ &= \frac{1}{M_1M_2}\sum_{w_1w_2}\underbrace{\sum_c p_C(c)p_{e,w_1w_2}^{(n)}(c)}_{\mathbb{E}p_{e,w_1w_2}^{(n)}(C)} \\ &= \mathbb{E}p_{e,11}^{(n)}(C), \quad (*)\end{aligned}$$

   where Eqn.(*) follows because $\mathbb{E}p_{e,w_1w_2}^{(n)}(C)$ does not depend on w_1w_2. Hence, we assume that $w_1w_2 = 11$, by symmetry of the random code construction.

   b) Error events: Define the event,

$$E_{ab} := \Big\{(q^n x_1^n(a)x_2^n(b)y^n) \in T_\delta^{(n)}\big(QX_1X_2Y\big)\Big\}.$$

   Let E denote the error event, i.e., $g(y^n) \neq 11$. Then,

$$\begin{aligned}E^c &= E_{11} \cap \bigcap_{b>1} E_{1b}^c \cap \bigcap_{a>1} E_{a1}^c \cap \bigcap_{\substack{a>1,\\ b>1}} E_{ab}^c \\ \text{and therefore, } E &= E_{11}^c \cup \bigcup_{b>1} E_{1b} \cup \bigcup_{a>1} E_{a1} \cup \bigcup_{\substack{a>1,\\ b>1}} E_{ab}\end{aligned}$$

   c) An observation: To evaluate probabilities of events $E_{11}, E_{1b}, E_{a1}, E_{ab}$, we observe that the corresponding random vector sequences have distribution:

$$\begin{array}{lcll} Q^nX_1^n(1)X_2^n(1)Y^n & \sim & p_{Q^n}p_{X_1^n|Q^n}p_{X_2^n|Q^n}p_{Y^n|X_1^nX_2^n} & \text{(true distribution)} \\ Q^nX_1^n(1)X_2^n(b)Y^n & \sim & p_{Q^n}p_{X_1^n|Q^n}p_{X_2^n|Q^n}p_{Y^n|X_1^nQ^n} & (X_2^n(b) \text{ does not determine } Y^n) \\ Q^nX_1^n(a)X_2^n(1)Y^n & \sim & p_{Q^n}p_{X_1^n|Q^n}p_{X_2^n|Q^n}p_{Y^n|X_2^nQ^n} & (X_1^n(a) \text{ does not determine } Y^n) \\ Q^nX_1^n(a)X_2^n(b)Y^n & \sim & p_{Q^n}p_{X_1^n|Q^n}p_{X_2^n|Q^n}p_{Y^n|Q^n} & (\text{Both } X_1^n(a), X_2^n(b) \text{ do not determine } Y^n) \end{array}$$

   - Typicality is w.r.t. joint distribution $p_Qp_{X_1|Q}p_{X_2|Q}p_{Y|X_1X_2} = p_Qp_{X_1|Q}p_{X_2|Q}p_{Y|X_1X_2Q}$.

We look at frequency typicality in Lecture 3 and complete the proof in Lecture 4. ■