E2–301 Topics in Multiuser Communication August 28, 2007

Lecture 3 : Frequency typicality

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1 Definitions

- $[m] := \{1, 2, 3, \cdots, m\}$
- $Z_{[m]} := (Z_1\ Z_2\ Z_3\ \cdots\ Z_m)$ a random vector taking values in $\mathbb{Z}_1 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_m =: \mathbb{Z}_{[m]}$
- For any $A \subseteq [m], Z_A := (Z_k : k \in A) \in \bigtimes_{k \in A} \mathbb{Z}_k =: \mathbb{Z}_A$
- $Z^n_{[m]} := (Z_{[m],1}\ Z_{[m],2}\ \cdots\ Z_{[m],n}), \quad Z_{[m],i} \in \mathbb{Z}_{[m]}$
- $Z^n_A := (Z_{A,1}\ Z_{A,2}\ \cdots\ Z_{A,n}),$ where $Z_{A,i} = (Z_{k,i} : k \in A), \quad Z_{A,i} \in \mathbb{Z}_A$
- $z^n_{[m]} \in \mathbb{Z}^n_{[m]} = \bigtimes_{i=1}^{n} \mathbb{Z}_{[m]}$
- $z^n_A \in \mathbb{Z}^n_A = \bigtimes_{i=1}^{n} \mathbb{Z}_A$
- $N(a_{[m]} \big| z^n_{[m]}) := \sum_{i=1}^n 1\left\{z_{[m],i} = a_{[m]}\right\}$ = frequency of occurrence of the m–tuple $a_{[m]}$, where $a_{[m]} \in \mathbb{Z}_{[m]}$, in the given sequence $z^n_{[m]}$.
- $T^{(n)}_\delta := \left\{ z^n_{[m]} \in \mathbb{Z}^n_{[m]} \ : \ \left| \frac{1}{n} N(a_{[m]} \big| z^n_{[m]}) - p_{Z_{[m]}}(a_{[m]}) \right| \leqslant \frac{\delta \ p_{Z_{[m]}}(a_{[m]})}{\log |\mathbb{Z}_{[m]}|}, \ \forall a_{[m]} \in \mathbb{Z}_{[m]} \right\}$
- $T^{(n)}_\delta(Z_A) := \left\{ z^n_A \in \mathbb{Z}^n_A \ : \ (z^n_A,\ z^n_{A^c}) \in T^{(n)}_\delta, \text{ for some } z^n_{A^c} \in \mathbb{Z}^n_{A^c} \right\}$. $T^{(n)}_\delta(Z_A)$ is the projection of $T^{(n)}_\delta$ on $\mathbb{Z}^n_A$.
- For any $z^n_B \in \mathbb{Z}^n_B$, $T^{(n)}_\delta(Z_A \big| z^n_B) := \left\{ z^n_A \in \mathbb{Z}^n_A \ : \ (z^n_A,\ z^n_B) \in T^{(n)}_\delta(Z_{A \cup B}) \right\}$. This is the projection of those sequences in $T^{(n)}_\delta$ with a specific component in B.
- $\left| \frac{1}{n} \log a_n - b \right| \leqslant \epsilon \iff a_n \stackrel{\circ}{=} 2^{nb \pm n\epsilon}$

Remarks:

1. $z^n_{[m]} \in T^{(n)}_\delta \implies z^n_A \in T^{(n)}_\delta(Z_A)$, for any $A \subseteq [m]$ (by definition).
2. $N(a_A \big| z^n_A) := \sum_{i=1}^{n} 1\left\{z_{A,i} = a_A\right\} = \sum_{a_{A^c}} N\left(a_A, a_{A^c} \big| z^n_A, z^n_{A^c}\right)$, for any arbitrary $z^n_{A^c} \in \mathbb{Z}^n_{A^c}$.
3. $z^n_{[m]} \in T^{(n)}_\delta \implies \left| \frac{1}{n} N(a_A \big| z^n_A) - p_{Z_A}(a_A) \right| \leqslant \frac{\delta \ p_{Z_A}(a_A)}{\log |\mathbb{Z}_{[m]}|}$. Indeed,

$$\begin{aligned}
\left| \frac{1}{n} N\left(a_A | z^n_A\right) - p_{Z_A}(a_A) \right| &= \left| \sum_{a_{A^c}} \frac{1}{n} N\left(a_A, a_{A^c} | z^n_A, z^n_{A^c}\right) - \sum_{a_{A^c}} p_{Z_{[m]}}(a_A, a_{A^c}) \right| \\
&\leqslant \sum_{a_{A^c}} \left| \frac{1}{n} N\left(a_A, a_{A^c} | z^n_A, z^n_{A^c}\right) - p_{Z_{[m]}}(a_A, a_{A^c}) \right| \\
&\leqslant \sum_{a_{A^c}} \delta \frac{p_{Z_{[m]}}(a_A, a_{A^c})}{\log |\mathbb{Z}_{[m]}|} \\
&= \delta \frac{p_{Z_A}(a_A)}{\log |\mathbb{Z}_{[m]}|}.
\end{aligned}$$

2 Frequency typicality

Lemma 1 *For every $\delta > 0$, the following hold for all sufficiently large n.*

1. $Pr\left\{Z^n_{[m]} \in T^{(n)}_\delta\right\} \geqslant 1-\delta$ *and therefore* $Pr\left\{Z^n_A \in T^{(n)}_\delta(Z_A)\right\} \geqslant 1-\delta$.

2. $z^n_A \in T^{(n)}_\delta(Z_A) \implies \left|\frac{1}{n}\log p_{Z^n_A}(z^n_A) + H(Z_A)\right| \leqslant \delta$.

3. $(z^n_A, z^n_B) \in T^{(n)}_\delta(Z_{A\cup B}),\quad A\cap B = \emptyset \implies \left|\frac{1}{n}\log p_{Z^n_A|Z^n_B}(z^n_A|z^n_B) + H(Z_A|Z_B)\right| \leqslant 2\delta$.

4. $(1-\delta)2^{nH(Z_A)-n\delta} \leqslant \left|T^{(n)}_\delta(Z_A)\right| \leqslant 2^{nH(Z_A)+n\delta}$ *so that* $\left|T^{(n)}_\delta(Z_A)\right| \doteq 2^{nH(Z_A)\pm 2n\delta}$.

5. $\widetilde{Z}_{[m]} \sim p_{Z_A}p_{Z_B|Z_A}p_{Z_C|Z_A}$, $A\cup B\cup C = [m], A\cap B = B\cap C = C\cap A = \emptyset, \widetilde{Z}^n_{[m]}$ *i.i.d. copies with generic distribution that of* $\widetilde{Z}_{[m]}$. *Then* $Pr\left\{\widetilde{Z}^n_{[m]} \in T^{(n)}_\delta\right\} \doteq 2^{-nI(Z_B;Z_C|Z_A)\pm 7n\delta}$.

Proof See solution to homework 1. ■

3 Conditional frequency typicality

Lemma 2 *For every $\delta > 0$, the following hold for all sufficiently large n.*

1. $z^n_A \in T^{(n)}_\delta(Z_A) \implies Pr\left\{Z^n_{A^c} \in T^{(n)}_{2\delta}(Z_{A^c}|z^n_A)\,\middle|\, Z^n_A = z^n_A\right\} \geqslant 1-\delta$, *so that for any* $B \subseteq A^c$, $Pr\left\{Z^n_B \in T^{(n)}_{2\delta}(Z_B|z^n_A)\,\middle|\, Z^n_A = z^n_A\right\} \geqslant 1-\delta$.

2. $z^n_A \in T^{(n)}_\delta(Z_A)$ *and* $B \subseteq A^c \implies (1-\delta)2^{nH(Z_B|Z_A)-2n\delta} \leqslant \left|T^{(n)}_{2\delta}(Z_B|z^n_A)\right| \leqslant 2^{nH(Z_B|Z_A)+2n\delta}$.

Proof See solution to homework 1. ■

Remarks: Note the 2δ in $T^{(n)}_{2\delta}(Z_B|z^n_A)$.