# Lecture 4 : Multiple Access Channels

*Instructor: Rajesh Sundaresan* *Scribe: Premkumar K.*

We need the following from the previous lecture: For any $\delta > 0$, the following hold for all sufficiently large $n$.

1. $\Pr\left\{Z^n_{[m]} \in T^{(n)}_\delta\right\} \geqslant 1-\delta$ and therefore $\Pr\left\{Z^n_A \in T^{(n)}_\delta(Z_A)\right\} \geqslant 1-\delta$.

2. $\widetilde{Z}_{[m]} \sim p_{Z_A}p_{Z_B|Z_A}p_{Z_C|Z_A}$, $A \cup B \cup C = [m], A\cap B = B \cap C = C \cap A = \emptyset, \widetilde{Z}^n_{[m]}$ i.i.d. copies with generic distribution that of $\widetilde{Z}_{[m]}$. Then, $\Pr\left\{\widetilde{Z}^n_{[m]} \in T^{(n)}_\delta\right\} \doteq 2^{-nI(Z_B;Z_C|Z_A)\pm 7n\delta}$.

## 1 Continuing with the proof on page 2 of lecture 2

In Lecture 2, we indicated the frequency typical set $T^{(n)}_\delta$. In the last lecture, we studied some properties of these sets. We now complete the proof of Proposition 3 (of Lecture 2).

- Since $E_{11} = \left\{q^n x^n_1(1) x^n_2(1) y^n \in T^{(n)}_\delta\right\}$, we have by Lemma 1.1, $\Pr\{E^c_{11}\} \leqslant \delta$.

  Since $\Pr\{E_{1b}\}$ is the same for all $b > 1$, we have

$$\begin{aligned}
\Pr\left\{\bigcup_{b>1} E_{1b}\right\} &\leqslant \sum_{b>1} \Pr\{E_{1b}\} \\
&= (M_2-1)\Pr\{E_{12}\} \\
&= (M_2-1)\Pr\left\{\widetilde{Z}^n \in T^{(n)}_\delta\right\} && A = QX_1,\ B = X_2,\ C = Y \text{ in Lemma 1.5} \\
&\leqslant (M_2-1)2^{-nI(X_2;Y|X_1,Q)+7n\delta} && \text{from Lemma 1.5} \\
&\leqslant 2^{n(R_2-\eta-I(X_2;Y|X_1,Q)+7\delta)} && \text{refer Eqn. 1 of Lecture 2} \\
&\underset{n}{\leqslant} \delta && \text{if } 7\delta < \eta
\end{aligned}$$

  Similarily

$$\Pr\left\{\bigcup_{a>1} E_{a1}\right\} \leqslant 2^{n(R_1-\eta-I(X_1;Y|X_2,Q)+7\delta)} \underset{n}{\leqslant} \delta$$

  and

$$\Pr\left\{\bigcup_{\substack{a>1\\b>1}} E_{ab}\right\} \leqslant 2^{n(R_1+R_2-\eta-I(X_1,X_2;Y|Q)+7\delta)} \underset{n}{\leqslant} \delta$$

  if $7\delta < \eta$. Therefore,

$$\Pr\{E\} \underset{n}{\leqslant} 4\delta \leqslant \lambda \ \text{ if } \delta < \lambda/4.$$

  Setting $\delta = \min\{\lambda/4,\ \eta/7\}$, completes the proof.

**Theorem 1** $\mathscr{C}_{MAC} = \mathscr{C}$

**Proof** Proposition 3 of Lecture 2 shows that $\mathscr{C} \subseteq \mathscr{C}_{\text{MAC}}$. It is sufficient to show the converse, i.e, $\mathscr{C}_{\text{MAC}} \subseteq \mathscr{C}$.

- Suppose $(R_1, R_2)$ is achievable, i.e., $(R_1, R_2) \in \mathscr{C}_{\text{MAC}}$. For an $\eta > 0, \lambda \in (0,1)$, consider a sequence of $(n, M_1, M_2)$ codes with

$$\left.\begin{array}{lrll} (1) & P_e^{(n)} \leqslant & \lambda & \\ (2) & \dfrac{\log M_k}{n} > & R_k - \eta, & k = 1, 2. \end{array}\right\} \quad \text{Eqn. (0)}$$

  where the inequalities hold for all sufficiently large $n$.

- Fix $n$. Consider the random vector sequence $W_1 W_2 X_1^n X_2^n Y^n$ induced by the code.

$$W_1 W_2 X_1^n X_2^n Y^n \sim p_{W_1}(w_1) p_{W_2}(w_2) p_{X_1^n|W_1}(x_1^n|w_1) p_{X_2^n|W_2}(x_2^n|w_2) p_{Y^n|X_1^n X_2^n}(y^n|x_1^n x_2^n)$$

$$\begin{aligned} p_{W_1}(w_1) &\sim \text{uniform on } \{1, 2, \cdots, M_1\} \\ p_{W_2}(w_2) &\sim \text{uniform on } \{1, 2, \cdots, M_2\} \\ p_{X_1^n|W_1}(x_1^n|w_1) &= \begin{cases} 1 & \text{if } x_1^n = f_1(w_1), \\ 0 & \text{if } x_1^n \neq f_1(w_1), \end{cases} \\ p_{X_2^n|W_2}(x_2^n|w_2) &= \begin{cases} 1 & \text{if } x_2^n = f_2(w_2), \\ 0 & \text{if } x_2^n \neq f_2(w_2), \end{cases} \\ p_{Y^n|X_1^n X_2^n}(y^n|x_1^n x_2^n) &= \prod_{i=1}^{n} p_{Y|X_1 X_2}(y_i|x_{1i}, x_{2i}) \end{aligned}$$

- Let $P_e^{(n)}(k)$ denote the average probability of error of user $k$. Clearly, $P_e^{(n)}(k) \leqslant P_e^{(n)} \leqslant \lambda$. Therefore, by Fano's inequality,

$$\left.\begin{array}{rcccl} H(W_1, W_2|Y^n) & \leqslant & (\log M_1 M_2) P_e^{(n)} + 1 & \leqslant & (\log M_1 M_2)\lambda + 1 \\ H(W_k|Y^n) & \leqslant & (\log M_k) P_e^{(n)}(k) + 1 & \leqslant & (\log M_k)\lambda + 1 \end{array}\right\} \quad \text{Eqn (1).}$$

- Moreover,

$$\left.\begin{array}{rll} H(W_1, W_2|Y^n) = H(W_1 W_2) - I(W_1 W_2; Y^n) & = \log M_1 M_2 - I(W_1 W_2; Y^n) \\ H(W_k|Y^n) = H(W_k) - I(W_k; Y^n) & = \log M_k - I(W_k; Y^n) \end{array}\right\} \quad \text{Eqn. (2)}$$

- Substitution of Eqn. (2) in Eqn. (1) yields,

$$\begin{aligned} (1-\lambda) \log M_1 M_2 &\leqslant I(W_1 W_2; Y^n) + 1 \\ (1-\lambda) \log M_1 &\leqslant I(W_1; Y^n) + 1 \\ (1-\lambda) \log M_2 &\leqslant I(W_2; Y^n) + 1. \end{aligned}$$

- Using Eqn. (0), we get

$$\begin{aligned}(R_1+R_2) &\leqslant \frac{1}{n(1-\lambda)} I(W_1W_2;Y^n) + 2\eta + \frac{1}{n(1-\lambda)} \\ &= \frac{1}{n} I(W_1W_2;Y^n) + \frac{\lambda}{n(1-\lambda)} I(W_1W_2;Y^n) + 2\eta + \frac{1}{n(1-\lambda)} \\ &\leqslant \frac{1}{n} I(W_1W_2;Y^n) + \frac{\lambda}{n(1-\lambda)} \log|\mathbb{Y}| + 2\eta + \frac{1}{n(1-\lambda)} \\ &\leqslant \frac{1}{n} I(W_1W_2;Y^n) + \epsilon \qquad \text{Eqn. (3)}\end{aligned}$$

  where Eqn. (3) holds for an arbitrary $\epsilon$ by choosing $\eta$ and $\lambda$ small enough.

- Similarly,

$$R_k \quad \leqslant \quad \frac{1}{n} I(W_1W_2;Y^n) + \frac{\lambda}{n(1-\lambda)} \log|\mathbb{Y}| + \eta + \frac{1}{n(1-\lambda)}, \quad k=1,2 \qquad \text{Eqn. (4)}$$

Before we proceed further, we need the following two lemmas.

**Lemma 2** *Consider $(A^n, B^n)$. Given $A_i$, the random variable $B_i$ is independent of all other variables, for each $i = 1, 2, \cdots, n$. Then,*

$$I(A^n; B^n) \leq \sum_{i=1}^{n} I(A_i; B_i)$$

*with equality if and only if $Y_1,\ Y_2,\ \cdots, Y_n$ are independent.*

**Proof** See solution to Homework 1. ■

**Corollary**: Let $W_1W_2X_1^nX_2^nY^n \sim p_{W_1}p_{W_2}p_{X_1^n|W_1}p_{X_2^n|W_2}p_{Y^n|X_1^nX_2^n}$ such that $p_{Y^n|X_1^nX_2^n}$ satisfies Eqn. (2) of Lecture 1.

$$\begin{aligned} I(W_1;Y^n) &\leqslant \sum_{i=1}^{n} I(X_{1i};Y_i|X_{2i}) \\ I(W_2;Y^n) &\leqslant \sum_{i=1}^{n} I(X_{2i};Y_i|X_{1i}) \\ I(W_1W_2;Y^n) &\leqslant \sum_{i=1}^{n} I(X_{1i}X_{2i};Y_i) \end{aligned}$$

**Proof** Observe that

$$\begin{aligned} W_1 &\longrightarrow X_1^n \longrightarrow Y^n \\ W_2 &\longrightarrow X_2^n \longrightarrow Y^n \\ (W_1, W_2) &\longrightarrow X_1^nX_2^n \longrightarrow Y^n \end{aligned}$$

$$
\begin{aligned}
\text{Thus } I(W_1;Y^n) &\leqslant I(X_1^n;Y^n) && \text{(data processing)}\\
&\leqslant I(X_1^n;Y^n X_2^n) && \text{(MI is nonnegative, chain rule)}\\
&= I(X_1^n;Y^n|X_2^n) && \text{($X_1^n$ is independent of $X_2^n$)}\\
&= \sum_{x_2^n \in \mathscr{X}_2^n} I(X_1^n;Y^n|X_2^n = x_2^n) p_{X_2^n}(x_2^n) &&\\
&\leqslant \sum_{x_2^n \in \mathscr{X}_2^n} p_{X_2^n}(x_2^n) \left[\sum_{i=1}^n I(X_{1i};Y_i|X_2^n = x_2^n)\right] && \text{(Lemma 2, for a fixed $x_2^n$)}\\
&= \sum_{x_2^n \in \mathscr{X}_2^n} p_{X_2^n}(x_2^n) \left[\sum_{i=1}^n I(X_{1i};Y_i|X_{2i} = x_{2i})\right] &&\\
&= \sum_{i=1}^n I(X_{1i};Y_i|X_{2i}) &&
\end{aligned}
$$

Others follow analogously. ■

- An application of the above corollary to Eqn. (3) and Eqn. (4) yields

$$
\begin{aligned}
(R_1 + R_2) &\leqslant \frac{1}{n}\sum_{i=1}^n I(X_{1i}X_{2i};Y_i) + \epsilon\\
R_1 &\leqslant \frac{1}{n}\sum_{i=1}^n I(X_{1i};Y_i|X_{2i}) + \epsilon\\
R_2 &\leqslant \frac{1}{n}\sum_{i=1}^n I(X_{2i};Y_i|X_{1i}) + \epsilon
\end{aligned}
$$

- We claim that

$$
\left.\begin{aligned}
\frac{1}{n}\sum_{i=1}^n I(X_{1i};Y_i|X_{2i}) &= I(\overline{X}_1;\overline{Y}|\overline{X}_2 Q)\\
\frac{1}{n}\sum_{i=1}^n I(X_{2i};Y_i|X_{1i}) &= I(\overline{X}_2;\overline{Y}|\overline{X}_1 Q)\\
\frac{1}{n}\sum_{i=1}^n I(X_{1i}X_{2i};Y_i) &= I(\overline{X}_1,\overline{X}_2;\overline{Y}|Q)
\end{aligned}\right\} \quad \text{Eqn. (5)}
$$

  for a $Z = Q\overline{X}_1\overline{X}_2\overline{Y} \in \mathscr{P}^*$, so that $(R_1 - \epsilon, R_2 - \epsilon) \in \mathscr{C}$. Since $\epsilon$ is arbitrary, $\mathscr{C}$ is closed, we have $(R_1, R_2) \in \mathscr{C}$.

- To verify the claim in Eqn. (5),
  - Let $\mathbb{Q} = \{1, 2, \cdots, n\}$ and define $Z = Q\overline{X}_1\overline{X}_2\overline{Y}$ via

$$
\Pr\left\{Q = i, \overline{X}_1 = a_1, \overline{X}_2 = a_2, \overline{Y} = b\right\} := \frac{1}{n}\Pr\left\{X_{1i}(W_1) = a_1, X_{2i}(W_2) = a_2, Y_i = b\right\}
$$

    Clearly, $\Pr\{Q = i\} = \frac{1}{n}$, so that Eqn. (5) holds. We need to show $Z \in \mathscr{P}^*$. Observe that

$$
p_{\overline{X}_1,\overline{X}_2,\overline{Y}|Q}(a_1 a_2 b|i) = \Pr\left\{X_{1i}(W_1) = a_1, X_{2i}(W_2) = a_2, Y_i = b\right\}
$$

so that

$$
\begin{aligned}
p_{\overline{X}_1,\overline{X}_2|Q}(a_1a_2|i) &= \Pr\{X_{1i}(W_1)=a_1, X_{2i}(W_2)=a_2\} \\
&= \Pr\{X_{1i}(W_1)=a_1\}\Pr\{X_{2i}(W_2)=a_2\} \quad \text{by independence of } W_1 \text{ and } W_2 \\
&= p_{\overline{X}_1|Q}(a_1|i)p_{\overline{X}_2|Q}(a_2|i)
\end{aligned}
$$

Next,

$$
\begin{aligned}
p_{\overline{Y}|Q\overline{X}_1,\overline{X}_2}(b|ia_1a_2) &= \Pr\{Y_i=b|X_{1i}(W_1)=a_1, X_{2i}(W_2)=a_2\} \\
&= p_{Y|X_1,X_2}(b|a_1a_2) \quad \text{(does not depend on } i\text{)}.
\end{aligned}
$$

So $Z \in \mathscr{P}^*$. This completes the proof. ■

We now take a closer look at the time–sharing variable and time–sharing.

**Definition 3** $\mathscr{D}$

- $\mathscr{P} := \left\{ Z = QX_1X_2Y : Z \in \mathscr{P}^*, \text{ but } |\mathbb{Q}| = 1 \right\}$. *Note that we may then write* $I(X_1;Y|X_2Q) = I(X_1;Y|X_2)$, *and so on.*
- $\mathscr{D} :=$ *closure conv* $\left( \bigcup_{Z \in \mathscr{P}} \mathscr{C}(Z) \right)$, *where* $\mathscr{C}(Z)$ *is as in Lecture 2.*

**Remark**

- We first identify the polyhedrons $\mathscr{C}(Z)$, take union, then take convex hull, and finally its closure to get $\mathscr{D}$.
- Recall that $\mathscr{C}$ does not have the conv operation, and that for $Z \in \mathscr{P}^*$,

$$
\mathscr{C}(Z) = \left\{ \begin{array}{ll} (R_1,R_2): & 0 \leqslant R_1 \leqslant I(X_1;Y|X_2Q), \\ & 0 \leqslant R_2 \leqslant I(X_2;Y|X_1Q), \\ & R_1+R_2 \leqslant I(X_1X_2;Y|Q) \end{array} \right\}.
$$

  i.e., we first identify the upper bounds

$$
\begin{gathered}
I(X_1;Y|X_2Q=q), \\
I(X_2;Y|X_1Q=q), \\
I(X_1X_2;Y|Q=q).
\end{gathered}
$$

  for each $q$, and then take their convex combination of the upper bounds via the distribution $p_Q$ to obtain the upper bounds

$$
\begin{gathered}
I(X_1;Y|X_2Q), \\
I(X_2;Y|X_1Q), \\
I(X_1X_2;Y|Q)
\end{gathered}
$$

  that define the polyhedron $\mathscr{C}(Z)$.
- $\mathscr{C}$ and $\mathscr{D}$ may possibly differ as illustrated by the following example.

**Example 1** *(***Cover and Thomas***, pp. 534–535)*

$$F_1 = \left\{ \begin{array}{ll} (r_1, r_2): & 0 \leqslant r_1 \leqslant 10, \\ & 0 \leqslant r_2 \leqslant 10, \\ & 0 \leqslant r_1 + r_2 \leqslant 100 \end{array} \right\},$$

$$F_2 = \left\{ \begin{array}{ll} (r_1, r_2): & 0 \leqslant r_1 \leqslant 20, \\ & 0 \leqslant r_2 \leqslant 20, \\ & 0 \leqslant r_1 + r_2 \leqslant 20 \end{array} \right\},$$

*So any point in the closure conv* $(F_1 \cup F_2)$ *satisfies* $r_1 + r_2 \leqslant 20$*. On the other hand* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *combination of constraints gives*

$$F = \left\{ \begin{array}{ll} (r_1, r_2): & 0 \leqslant r_1 \leqslant 15, \\ & 0 \leqslant r_2 \leqslant 15, \\ & 0 \leqslant r_1 + r_2 \leqslant 60 \end{array} \right\}.$$

*Clearly* $(15, 15) \in F$*, but does not belong to closure conv* $(F_1 \cup F_2)$*.*

**Remark**

- In general, we anticipate $\mathscr{C}$ is larger than $\mathscr{D}$.
- The property $I(X_1 X_2; Y) \leqslant I(X_1; Y|X_2) + I(X_2; Y|X_1)$ enables us to say $\mathscr{C} = \mathscr{D}$.