E2–301 Topics in Multiuser Communication	August 30, 2007
Lecture 5 : Polymatroids	
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Definition 1 *Let $r \in \mathbb{R}_+^K$, $S \subseteq [K]$. Then $r(S) := \sum_{k \in S} r_k$.*

Definition 2 (Set function)

$$f : 2^{[K]} \longrightarrow \mathbb{R}_+$$
$$S \longmapsto f(S)$$

Remark: For a given $r \in \mathbb{R}_+^K$, the mapping $r(\cdot)$ is a set function.

Definition 3 (Polyhedron) $\mathcal{B}(f) := \left\{ r \in \mathbb{R}_+^K : r(S) \le f(S), \forall S \subseteq [K] \right\}$

Definition 4 (Rank function) *A set function f is a rank function if it satisfies*

- *(**Normalised**)* $f(\emptyset) = 0$
- *(**Non–decreasing**)* $S \subseteq T \implies f(S) \le f(T)$
- *(**Submodular**)* $f(S \cup T) + f(S \cap T) \le f(S) + f(T)$.

Remark: $r(\cdot)$ above is a rank function.

Definition 5 (Polymatroid) *$([K], f)$ is a polymatroid if f is a rank function. Its associated polyhedron is $\mathcal{B}(f)$.*

Definition 6 *Consider the object $([K], f)$. For each permutation π on $[K]$, define $v(\pi) \in \mathbb{R}_+^K$ as follows:*

- $\big(v(\pi)\big)_{\pi(1)} = f\left(\{\pi(1)\}\right)$
- $\big(v(\pi)\big)_{\pi(k)} = f\big(\{\pi(1), \pi(2), \ldots, \pi(k)\}\big) - f\big(\{\pi(1), \pi(2), \ldots, \pi(k-1)\}\big), \; k = 2, 3, \cdots, n$

Remark: It may be the case that $v(\pi) \notin \mathcal{B}(f)$.

Definition 7 *$r \in \mathcal{B}(f)$ is maximal if $r' \geqslant r$ component–wise, $r' \in \mathcal{B}(f) \implies r' = r$.*

Definition 8 *$r \in \mathcal{B}(f)$ is extreme if $r = r'\lambda + r''(1-\lambda)$, $\lambda \in [0,1]$, $r', r'' \in \mathcal{B}(f) \implies \lambda = 0$ or 1.*

Definition 9 *$a, b \in \mathbb{R}_+^K$. a dominates b if $a_k \geqslant b_k$ for $k = 1, 2, \cdots, K$.*

Example 1 *($\pi = \sigma$, $\pi(i) = i$.)*

$$v(\sigma) = \Big(f(\{1\}), f(\{1,2\}) - f(\{1\}), \cdots, f(\{1,2,\cdots,K\}) - f(\{1,2,\cdots,K-1\}) \Big)$$

Facts ([Edmonds 1970]):

1. Let $([K], f)$ be a polymatroid. Then
   - r is a maximal extremal point of $\mathcal{B}(f) \iff r = v(\pi)$ for some π.
   - $r \in \mathcal{B}(f) \implies r$ is dominated by a convex combination of the maximal extremal points.
2. If $v(\pi) \in \mathcal{B}(f)$ for every π, then f is a rank function and $([K], f)$ is a polymatroid.

3. If $([K], f)$ is a polymatroid, $\lambda \in \mathbb{R}_+^K$, then $\max_{r \in \mathcal{B}(f)} \lambda^t r$ is attained at $r^* = v(\pi^*)$ where π^* is any permutation that satisfies $\lambda_{\pi^*(1)} \geqslant \lambda_{\pi^*(2)} \geqslant \cdots \geqslant \lambda_{\pi^*(K)}$.

Remarks:

- There are at most $K!$ maximal extreme points.
- (2) gives a way to check if $\mathcal{B}(f)$ is the polyhedron of a polymatroid.
- (3) solves a linear program with $2^K - 1$ constraints in a greedy fashion. Assignment to $\pi(k)$ tightens constraints, $k = 1, 2, 3, \cdots, n$.

Theorem 10 *Let $(X_1\ X_2\ \cdots\ X_K)$ be independent random variables. For the random vector $(X_1\ X_2\ \cdots\ X_K\ Y)$ and any $S \subseteq [K]$, the function $\rho : 2^{[K]} \longrightarrow \mathbb{R}_+$ defined by*

$$\rho(S) \quad := \quad \begin{cases} I(X_S; Y | X_{S^c}) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset \end{cases}$$

is a (submodular) rank function.

Proof

(i) ρ is normalised (by definition).

(ii) Monotonicity : $S \subseteq T, R = T - S$.

$$\begin{aligned} \rho(T) &= I(X_T; Y | X_{T^c}) \\ &= I(X_S X_R; Y | X_{T^c}) \\ &= I(X_R; Y | X_{T^c}) + I(X_S; Y | X_{T^c} X_R) \\ &\geqslant I(X_S; Y | X_{S^c}) \\ &= \rho(S) \end{aligned}$$

(iii) To show submodularity, we need to show,
$I(X_{S \cup T}; Y | X_{S^c \cap T^c}) + I(X_{S \cap T}; Y | X_{S^c \cup T^c}) \leqslant I(X_S; Y | X_{S^c}) I(X_T; Y | X_{T^c})$.
By independence of X_ks, it is sufficient to show

$$\begin{aligned} & H(X_{S \cup T}) + H(X_{S \cap T}) - H(X_{S \cup T} | Y X_{S^c \cap T^c}) - H(X_{S \cap T} | Y X_{S^c \cup T^c}) \\ \leqslant \;& H(X_S) + H(X_T) - H(X_S | Y X_{S^c}) - H(X_T | Y X_{T^c}). \end{aligned} \tag{1}$$

Observe that

$$H(X_{S \cup T}) + H(X_{S \cap T}) \quad = \quad H(X_S) + H(X_T) \tag{2}$$

by independence and because elements in $S \cap T$ occur twice on each side. Furthermore:

$$\begin{aligned} H(X_T | Y X_{T^c}) &= H(X_{T-S} | Y X_{T^c}) + H(X_{S \cap T} | Y X_{T^c \cup (T-S)}) \\ &= H(X_{T-S} | Y X_{T^c}) + H(X_{S \cap T} | Y X_{(S \cap T)^c}) & (3) \\ \text{and } H(X_{S \cup T} | Y X_{S^c \cap T^c}) &= H(X_{T-S} | Y X_{S^c \cap T^c}) + \underbrace{H(X_S | Y X_{S^c \cap T^c \cup (T-S)})}_{H(X_S | Y X_{S^c})} & (4) \end{aligned}$$

Substitution of the above in Eqn. 1 yields that it is sufficient to show

$$\begin{aligned} -H(X_{T-S}|YX_{S^c\cap T^c}) &\leqslant -H(X_{T-S}|YX_{T^c}) \\ \text{or} \qquad H(X_{T-S}|YX_{T^c}) &\leqslant H(X_{T-S}|YX_{S^c\cap T^c}) \end{aligned} \tag{5}$$

But Eqn. 5 holds because conditioning reduces entropy. ■

Corollary: Consider $QX_1 \cdots X_K Y$, where Q takes values in an arbitrary set $\mathbb{Q}$, and $X_1 X_2 \cdots X_K$ are independent given Q. Then,

$$\rho(S) := \begin{cases} I(X_S; Y|X_{S^c}Q) & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset \end{cases}$$

is a (submodular) rank function. ■

Remark: Specifically for two users:

(0) $\mathscr{C}(Z)$ is a polyhedron associated with a polymatroid.

(1) $I(X_1; Y|X_2) + I(X_2; Y|X_1) \geqslant I(X_1 X_2; Y) + 0$ (submodularity).

(2) Has two maximal extreme points:

$$\left(I(X_1; Y|X_2), \underbrace{I(X_1X_2; Y) - I(X_1; Y|X_2)}_{I(X_2;Y)} \right)$$

$$\textit{and} \quad \left(\underbrace{I(X_1X_2; Y) - I(X_2; Y|X_1)}_{I(X_1;Y)}, I(X_2; Y|X_1) \right)$$

There are no other maximal extreme points (Fact 1).

(3) Any point in the polyhedron $\mathscr{C}(Z)$ is dominated by a convex combination of the maximal extreme points (Fact 1).