E2–301 Topics in Multiuser Communication September 06, 2007

Lecture 6 : Equivalence of $\mathscr{C}$ and $\mathscr{D}$, and Caratheodory's theorem

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1 $\mathscr{C} = \mathscr{D}$

Theorem 1. $\mathscr{C} = \mathscr{D}$

Proof. $(\mathscr{D} \subseteq \mathscr{C})$

It is sufficient to show $\mathscr{D}_1 := \text{conv}\left(\bigcup_{Z\in\mathscr{P}} \mathscr{C}(Z)\right) \subseteq \mathscr{C}_1 := \left(\bigcup_{Z\in\mathscr{P}^*} \mathscr{C}(Z)\right)$

Let $R \in \mathscr{D}_1$. This implies $\exists L \in \mathbb{N}$, $\exists Z^{(\ell)} \in \mathscr{P}, \ell \in [L]$, $\exists R^{(\ell)} \in \mathscr{C}(Z^{(l)}), \ell \in [L]$, and $\exists \lambda_\ell \geqslant 0, \ell \in [L]$, such that $\sum_{\ell\in[L]} \lambda_\ell = 1$ and

$$R = \sum_{\ell\in[L]} \lambda_\ell R^{(\ell)}.$$

Since,

$$\begin{aligned} R^{(\ell)} &\in \mathscr{C}(Z^{(\ell)}), \quad \ell \in [L], \text{ we have} \\ \sum_{l\in S} R^{(\ell)} &\leqslant I(X_S^{(\ell)}; Y^{(\ell)} \Big| X_{S^c}^\ell), \quad \ell \in [L] \\ \text{and therefore, } \sum_{\ell\in[L]} \lambda_\ell R^{(\ell)} &\leqslant \sum_{\ell\in[L]} \lambda_\ell I(X_s^{(\ell)}; Y^{(\ell)} \Big| X_{S^c}^\ell), \ell \in [L] = I(X_S; Y \Big| X_{S^c} Q), \end{aligned}$$

for a suitably defined $Z = QX_1X_2Y \in \mathscr{P}^*$. Thus $R \in \mathscr{C}(Z)$ for some $Z \in \mathscr{P}^*$ and therefore $R \in \mathscr{C}_1$.

We now prove the other part: $\mathscr{C} \subseteq \mathscr{D}$. Once again, it is sufficient to show that $\mathscr{C}_1 \subseteq \mathscr{D}_1$.

Let $R \in \mathscr{C}_1$, i.e., $R \in \mathscr{C}(Z)$ for some $Z \in \mathscr{P}^*$.

$\mathscr{C}(Z)$ is a polyhedron associated with a polymatroid.

By Edmonds' result, R is dominated by a convex combination of the maximal extreme points of $\mathscr{C}(Z)$.

We show that every maximal extreme point of $\mathscr{C}(Z)$ is in $\mathscr{D}_1$ to complete the proof that $R \in \mathscr{D}$. Let $r \in \mathscr{C}(Z)$ be a maximal extreme point. By Edmonds' result, refer to fact in Lec. 5, r is a $v(\pi)$ for some permutation π, i.e.,

$$r_{k_i} = \rho\left(\{k_1, k_2, \cdots, k_i\}\right) - \rho\left(\{k_1, k_2, \cdots, k_{i-1}\}\right), \quad i = 1, 2, \cdots, K$$

where $k_1, k_2, \cdots, k_K$ is some permutation of $[K]$. Expanding r_{k_i}, we get

$$\begin{aligned} r_{k_i} &= I(X_{k_1}, X_{k_2}, \cdots, X_{k_i}; Y|X_{k_{i+1}}, \cdots, X_{k_K}, Q) - I(X_{k_1}, X_{k_2}, \cdots, X_{k_{i-1}}; Y|X_{k_i}, X_{k_{i+1}}, \cdots, X_{k_K}, Q) \\ &= \sum_{\ell=1}^{|\mathbb{Q}|} p_Q(\ell) \left[\rho_\ell\left(\Big\{k_1, k_2, \cdots, k_i\Big\}\right) - \rho_\ell\left(\Big\{k_1, k_2, \cdots, k_{i-1}\Big\}\right)\right] \end{aligned}$$

where $\rho_\ell(S) = I(X_S; Y \Big| X_{S^c}, Q = \ell)$. This implies that r is a convex combination of maximal extreme points of the polymatroidal polyhedra $\mathscr{C}(Z^{(\ell)})$, where $Z^{(\ell)} \in \mathscr{P}$ and therefore $r \in \mathscr{D}_1$.

□

2 Bounds on $|\mathbb{Q}|$

Recall that $\mathscr{C} = \text{closure}\left(\bigcup_{Z\in\mathscr{P}^*} \mathscr{C}(Z)\right)$, where $Z = QX_1X_2Y$.

Theorem 2. *(**Caratheodory**) If $A \subseteq \mathbb{R}^d$ and $a^* \in$ conv A, then $a^* = \sum_{\ell=0}^{d} \lambda_\ell a^{(\ell)}$, where $a^{(\ell)} \in A$ and $\sum_{\ell=0}^{d} \lambda_\ell = 1, \lambda_\ell \geqslant 0, \quad \forall \ell \in [d]$.*

Proof. Exercise. See Grunbaum for an elegant proof. □

Theorem 3. *$\mathscr{C}$ does not reduce if we restrict $Z = QX_1X_2Y \in \mathscr{P}^*$ to those vectors such that $|\mathbb{Q}| = 4$.*

Proof. Consider $Z = QX_1X_2Y \in \mathscr{P}^*$ with Q taking values in $\mathbb{Q} = \{1, 2, \cdots, |\mathbb{Q}|\}$.

- Observe that $X_1^{(\ell)}X_2^{(\ell)}Y^{(\ell)} \sim p_{X_1X_2Y|Q}(\cdot|Q=\ell) \in \mathscr{P}$.
- Also, if $\overline{\mathbb{Q}} \subseteq \mathbb{Q}$, $\overline{Q}$ any random variable taking values in $\overline{\mathbb{Q}}$, then $Z = \overline{Q}X_1X_2Y$ defined by $p_{\overline{Z}} = p_{\overline{Q}X_1X_2Y}(\ell x_1x_2y) = p_{\overline{Q}}(\ell)p_{X_1X_2Y|Q}(\cdot|Q=\ell) \in \mathscr{P}^*$.
- $\mathscr{C}(Z)$ is completely defined by

$$a = \begin{bmatrix} I(X_1;Y|X_2Q) \\ I(X_2;Y|X_1Q) \\ I(X_1X_2;Y|Q) \end{bmatrix} \in \text{conv } A,$$

  where

$$A = \left\{ a^{(\ell)} = \begin{bmatrix} I(X_1;Y|X_2, Q=\ell) \\ I(X_2;Y|X_1, Q=\ell) \\ I(X_1X_2;Y|Q=\ell) \end{bmatrix}, \quad \ell = 1, 2, \cdots, |\mathbb{Q}| \right\} \subseteq \mathbb{R}^3.$$

- By Caratheodory's theorem, $\exists \overline{Q} = \{\ell_0, \ell_1, \ell_2, \ell_3\} \subseteq \mathbb{Q}$ such that $a = \sum_{m=0}^{3} \lambda_{\ell_m} a^{(\ell_m)}$
- Define $\overline{Q}$ as follows: $p_{\overline{Q}}(\ell_m) = \lambda_{\ell_m}, \; m = 0, 1, 2, 3$, to get $\overline{Z} = \overline{Q}X_1X_2Y \in \mathscr{P}^*$.
- Easy to extend the above argument to K users, in which case we need $|\mathbb{Q}| = 2^K$.

□