E2–301 Topics in Multiuser Communication September 12, 2007

Lecture 7 : Han's generalisations

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1 Slepian–Wolf–Han Generalisation – Correlated Sources

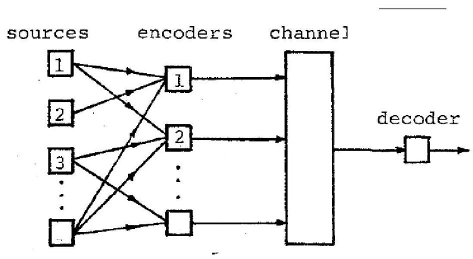

Figure 1: Generalised Multiple Access Channel with Correlated Sources.

- Define $\partial_k^- := \{j \in [J] : \exists$ a connection between user k and message source $j\}$.
- Connections are specified via $\partial_k^- \subseteq [J], k \in [K]$. Alternatively , they are via $\partial_j^+ \subseteq [K], j \in [J]$. Let us call this system $\mathsf{MAC}(J, K, 1)$ with dependence on ∂^- and $p_{Y|X_{[K]}}$ suppressed.
- $\mathscr{P} := \left\{ Z = U_{[J]} X_{[K]} Y \text{ that satisfy (1)–(5) below} \right\}$:

(1) U_j takes values in an arbitrary finite set $\mathbb{U}_j$, $j \in [J]$;
(2) X_k takes values in the given finite set $\mathbb{X}_k$, $k \in [K]$;
(3) Y takes values in the given finite set $\mathbb{Y}$;
(4) $\exists f_k$ such that $f_k : \underbrace{\prod_{j \in \partial_k^-} \mathbb{U}_j}_{\mathbb{U}_k} \to \mathbb{X}_k$, $k \in [K]$,

satisfying $p_{X_k|U_{[J]}}\Big(x_k|u_{[J]}\Big) = p_{X_k|U_{\partial_k^-}}\Big(x_k|u_{\partial_k^-}\Big) = 1\Big\{x_k = f_k\Big(u_{\partial_k^-}\Big)\Big\}$

(5)

$$p_{U_{[J]} X_{[K]} Y} = \left(\prod_{j \in [J]} p_{U_j} \right) \left(\prod_{k \in [K]} p_{X_k|U_{\partial_k^-}} \right) \Big(p_{Y|X_{[K]}} \Big).$$

Remark 1. $U_{[J]} \to X_{[K]} \to Y$.

- $\mathscr{P}^* := \left\{ Z = QU_{[J]} X_{[K]} Y \textit{ that satisfy (0)–(5) below} \right\}$:

(0) Q takes values in an arbitrary finite set $\mathbb{Q}$;
(1) – (4) as above, with f_k replaced by $f_k\Big(U_{\partial_k^-}|Q\Big)$;
(5)

$$p_{QU_{[J]} X_{[K]} Y} = p_Q \left(\prod_{j \in [J]} p_{U_j|Q} \right) \left(\prod_{k \in [K]} p_{X_k|U_{\partial_k^-} Q} \right) \Big(p_{Y|X_{[K]}} \Big).$$

- For $S \subseteq J$, define

$$\rho(S) \quad := \quad \begin{cases} 0, & \text{if } S = \emptyset \\ I(X_S; Y|QX_{S^c}) & \text{otherwise} \end{cases}$$

Note that $\rho(\cdot)$ is a submodular rank function.

- For $Z \in \mathscr{P}^*$, $\mathscr{C}(Z)$ is the polyhedron associated with the polymatroid $([J], \rho)$.
- Define $\mathscr{C}$ and $\mathscr{D}$ as before.

Theorem 1. *$\mathscr{C}_{MAC}(J, K, 1) = \mathscr{C} = \mathscr{D}$.*

Proof. Read Han.
No new elements except for a careful application of Fano's inequality, for each subset $S \subseteq [J]$. □

Remark 2. *Cardinality bounds for $\mathbb{Q}$, and $\mathbb{U}_j, j \in [J]$ require a generalization of Caratheodory's theorem. Read Han's paper for cardinality bounds on $\mathbb{U}_j$ when $|\mathbb{Q}| = 1$.*

2 Ahlswede–Ulrey–Han Generalisation – Multiple Output Terminals

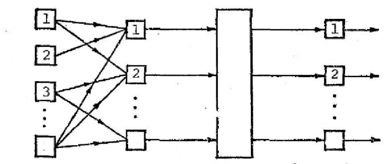

Figure 2: Multiple Access Channel with Correlated Sources and Multiple Output Terminals.

- Every output terminal is interested in all source symbols.
- One may view this as a compound channel. Two formulations possible: (1) $P_e^{(n)}(\ell) \leqslant \lambda, \forall \ell \in [L]$; (2) $P_e^{(n)} := \Pr\Big\{g_\ell(Y_\ell^n) \neq W_{[J]}, \text{ for some } \ell \in [L]\Big\} \leqslant \lambda$.
- **Remark 3.** *(A simplification) The capacity of MAC(J,K,L) with a general $p_{Y_{[L]}|X_{[K]}}$ is the same as the capacity of a MAC(J,K,L) with*

$$\overline{p}_{Y_{[L]}|X_{[K]}} \quad = \quad \prod_{\ell \in [L]} p_{Y_\ell|X_{[K]}}.$$

This is because, given Y_ℓ, $P_e^{(n)}(\ell)$ does not depend on $Y_{[L]\setminus\{\ell\}}$.

- **Remark 4.** *The channel capacity remains the same under the two formulations (because $\max_{\ell \in [L]} P_e^{(n)}(\ell) \leqslant P_e^{(n)} \leqslant \sum_{\ell \in [L]} P_e^{(n)}(\ell)$). So, we may restrict our attention to conditionally independent outputs.*

- We can therefore set $\mathscr{P}^* := \left\{ Z = QU_{[J]}X_{[K]}Y \text{ such that (0)–(5) below hold} \right\}$:

  (0) Q takes values in an arbitrary finite set $\mathbb{Q}$;

  (0) – (4) holds as above, where in (4) f_k is replaced by $f_k\left(U_{\partial_k^-}\middle|Q\right)$;

  (5)

$$p_{QU_{[J]}X_{[K]}Y} \quad = \quad p_Q \left(\prod_{j\in[J]} p_{U_j|Q}\right)\left(\prod_{k\in[K]} p_{X_k|U_{\partial_k^-}Q}\right)\left(\prod_{\ell\in[L]} p_{Y_\ell|X_{[K]}}\right).$$

- $\mathscr{C}(Z)$ is the polygon associated with $([K], \rho)$ where

$$\rho(S) \quad := \quad \min_{\ell\in[L]} \rho_\ell(S)$$

  and

$$\rho_\ell(S) \quad := \quad I(U_S; Y_\ell | U_{S^c}Q)$$

  Remark 5. *ρ is not necessarily a rank function. See Fig. 3 below.*

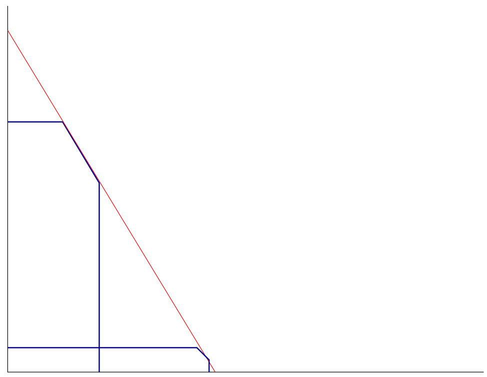

Figure 3: An example where $([2], \rho)$ is not a polymatroid.

Theorem 2. *(Han, 79)* $\mathscr{C}_{MAC}(J, K, L) = \textit{closure}\left(\bigcup_{Z\in\mathscr{P}^*} \mathscr{C}(Z)\right) = \mathscr{C}$.

Proof. Typicality at each terminal is with respect to $T_\delta^{(n)}\left(QU_{[J]}X_{[K]}Y_{[L]}\right)$. □

Remark 6. *Cardinality bounds apply.*

Note 1. *$\mathscr{D} \subsetneq \mathscr{C}$. There is an error in Thm. 5.1 of Han. See Remark 2.2 of Han-Kobayashi p.53.*

Example 1 (Gaussian). $Y = \underset{k\in[K]}{X_k} + W_k$.

- $\bigcup_{Z\in\mathscr{P}^*} \mathscr{C}(Z) = \mathscr{C}(Z_G)$, *where* $Z_G = X_{[K],G}Y$, $X_{[K],G} \sim N(0, diag(P_1, P_2, \cdots, P_K))$, *so that*
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$$\mathscr{C}_{MAC}(K, K, 1) \quad = \quad \left\{ R \in \mathbb{R}_+^K : R(S) \leqslant \frac{1}{2}\log\left(1 + \frac{P(S)}{\sigma^2}\right), \ \forall S \subseteq [K] \right\}.$$

Example 2 (Gaussian vector MAC with noise covariance $\sum$).

$$\mathscr{C}_{MAC}(K, K, 1) \quad = \quad \left\{ R \in \mathbb{R}_+^K : R(S) \leqslant \frac{1}{2}\log\left| I_N + \sum^{-1} sP(S)s^T \right|, \ \forall S \subseteq [K] \right\}$$

where $s = [s_1, s_2, \cdots, s_K], s_k \in \mathbb{R}_+^N$, *signature of user* k.