E2–301 Topics in Multiuser Communication	September 13, 2007

Lecture 8 : Interference Channels

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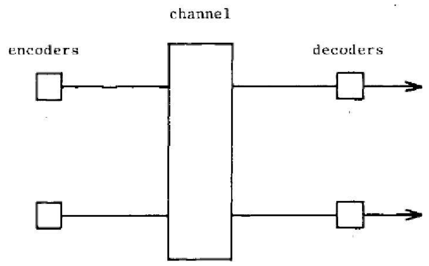

Figure 1: Interference Channel.

1 Definitions

Definition 1 (DM–IC). *A (two user) discrete memoryless interference channel (DM–IC) denoted by* $\left(\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}_1, \mathbb{Y}_2, p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)\right)$*, consists of finite sets* $\mathbb{X}_1, \mathbb{X}_2$*,* $\mathbb{Y}_1$*, and* $\mathbb{Y}_2$*, and a collection of probability mass functions* $p_{Y_1Y_2|X_1X_2}(\cdot|x_1x_2)$ *on* $\mathbb{Y}_1 \times \mathbb{Y}_2$*, one for each* $x_1x_2 \in \mathbb{X}_1 \times \mathbb{X}_2$*, with the interpretation that* X_k *is the input of user* k*,* $k = 1, 2$ *and* Y_k *is the input to the decoder of user* k*. For* $n \in \mathbb{N}$*, with* $X_k^n = (X_{k1}, X_{k2}, \cdots, X_{kn})$*,* $k = 1, 2$ *as inputs, the output sequence* $Y_1^n Y_2^n$ *has pmf*

$$p_{Y_1^nY_2^n|X_1^nX_2^n}(y_1^ny_2^n|x_1^nx_2^n) = \prod_{i=1}^{n} p_{Y_1Y_2|X_1X_2}(y_{1i}y_{2i}|x_{1i}x_{2i}) \quad (1)$$

Definition 2 (Code). *An* (n, M_1, M_2) *code for the channel* $\left(\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}_1, \mathbb{Y}_2, p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)\right)$ *consists of the following:*

1. *An index set of messages for each user* k*,* $\mathbb{W}_k = \{1, 2, \cdots, M_k\} = [M_k]$*.*
2. *An encoder* f_k *for each user* k*,* $f_k : [M_k] \to \mathbb{X}_k^n$*,* $k = 1, 2$*. Note that* $[M_k] \ni W_k \mapsto f_k(W_k) \in \mathbb{X}_k^n$*. The codebook can be represented by an ordered set*
$$c = \{f_1(1), f_1(2), \cdots, f_1(M_1); f_2(1), f_2(2), \cdots, f_2(M_2)\}.$$
3. *A decoding rule,* $g_k : \mathbb{Y}_k^n \to \phi \cup [M_k], k = 1, 2$*, i.e.,* $y_k^n \mapsto g_k(y_k^n) = \hat{w}_k \in \phi \cup [M_k]$*. Note that* g_k *partitions* $\mathbb{Y}_k^n$ *into decision regions.*

Definition 3 (Probability of error). *Let* W_k *be the message transmitted by user* k *and let* Y_k^n *be the signal received by user* k*. The conditional probability of error for user* k *when* $(W_1W_2) = (w_1w_2)$ *was transmitted is given by*

$$P_{e,w_1w_2}^{(n)}(c; k) = Pr\{g_k(Y_k^n) \neq W_k | W_1W_2 = w_1w_2\}, \; k = 1, 2.$$

The average probability of error for the code c *(for user* k*) is given by*

$$P_e^{(n)}(c; k) = \frac{1}{M_1M_2} \sum_{w_1w_2} P_{e,w_1w_2}^{(n)}(c; k) \quad k = 1, 2.$$

Note that the above equation assumes that all messages are equally likely and the users choose their messages independently.

Definition 4 (Achievability). *The rate pair (R_1, R_2) is achievable, if for every $\eta > 0, \lambda \in (0,1)$, there exists a sequence of (n, M_1, M_2) codes that satisfy*

1. $P_e^{(n)}(k) \leq \lambda$, *and*
2. $\frac{\log_2 M_k}{n} > R_k - \eta$, $k = 1, 2$

for all sufficiently large n.

Definition 5 (Capacity region). *The capacity region is the set of all achievable rate pairs, denoted by $\mathscr{C}_I$.*

Remark 1.

- $\mathscr{C}_I$ *is closed and convex.*
- $R \in \mathscr{C}_I$, R *dominates* $r \implies r \in \mathscr{C}_I$.
- $\mathscr{C}_I$ *depends on* $p_{Y_1Y_2|X_1X_2}$ *only through the marginals* $p_{Y_k|X_1X_2}$, $k = 1, 2$.

Example 1. *(**Gaussian interference channel. Caution:** $\mathbb{X}_k = \mathbb{R}$)*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{a_{12}} \\ \sqrt{a_{21}} & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

where $\xi_k \sim N(0,1)$; X_1 and X_2 are power constrained

$$\frac{1}{n}\sum_{i=1}^{n} x_{ki}^2(w_k) \leqslant P_k, \quad \forall w_k \in [M_k].$$

- $\mathscr{C}_I$ *for this channel is not fully known. But,*

Proposition 1. *For the Gaussian interference channel, if $a_{12} \gg 1$ and $a_{21} \gg 1$, then $\mathscr{C}_I = \mathscr{C}_{MAC(2,2,2)}$ is given by*

$$\mathscr{C}_I = \Bigg\{ R \in \mathbb{R}_+^2 : R_k \leqslant \frac{1}{2}\log\left(1 + P_k\right), k = 1, 2,$$
$$R_1 + R_2 \leqslant \min\left\{\frac{1}{2}\log\left(1 + P_1 + a_{12}p_2\right), \frac{1}{2}\log\left(1 + P_2 + a_{21}P_1\right)\right\}\Bigg\}$$

Remark 2.

- $\mathscr{C}_I$ *is not necessarily a polyhedron associated with a polymatroid.*
- $a_{12} \geqslant 1 + P_1$ *and* $a_{21} \geqslant 1 + P_2 \implies \mathscr{C}_I$ *is a rectangle; as if there were no interference.*

Proof. **Achievability**: Make both decoders decode both streams. Hence, $\mathscr{C}_{\text{MAC}(2,2,2)} \subseteq \mathscr{C}_{\text{I}}$.
Converse: If $a_{ij} \geqslant 1, i \neq j$, then the other user sees a stronger channel and should be able to decode the unwanted signal. Let $R \in \mathscr{C}_{\text{I}}$. For a fixed $\lambda/2 \in (0,1), \eta > 0$, consider a sequence of (n, M_1, M_2) codes that satisfy $P_e^n(k) \underset{n}{\leqslant} \lambda/2$ and the rate condition.
In vector notation, the received signals are

$$\begin{aligned} y_1^n &= x_1^n + \sqrt{a_{12}}\, x_2^n + \xi_1^n \\ y_2^n &= \sqrt{a_{21}}\, x_1^n + x_2^n + \xi_2^n. \end{aligned}$$

Also, $f_k\big(g_k\big(y_k^n\big)\big) = x_k^n$ with probability $\geqslant 1 - \lambda/2$. Consider,

$$\widehat{y}_1^n = \frac{y_2^n - f_2\Big(g_2\Big(y_2^n\Big)\Big)}{\sqrt{a_{21}}} + \sqrt{a_{12}} f_2\Big(g_2\Big(y_2^n\Big)\Big) + \sqrt{1 - \frac{1}{a_{21}}}\,\widehat{\xi}_1^n,$$

where $\widehat{\xi}_1^n$ is independent Gaussian noise generated at the receiver. Similarly obtain $\widehat{y}_2^n$. Define $G_1(y_1^n) := (g_1(y_1^n), g_2(\widehat{y}_2^n))$. Similarly define G_2. Define:

$$\begin{bmatrix} \widetilde{Y}_1^n \\ \widetilde{Y}_2^n \end{bmatrix} := \begin{bmatrix} 1 & \sqrt{a_{12}} \\ \sqrt{a_{21}} & 1 \end{bmatrix} \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} + \begin{bmatrix} \widetilde{\xi}_1^n \\ \widetilde{\xi}_2^n \end{bmatrix},$$

where $\widetilde{\xi}_1^n = \frac{\xi_2^n}{\sqrt{a_{21}}} + \sqrt{1 - \frac{1}{a_{21}}}\widehat{\xi}_1^n$ and $\widetilde{\xi}_2^n = \frac{\xi_1^n}{\sqrt{a_{12}}} + \sqrt{1 - \frac{1}{a_{12}}}\widehat{\xi}_2^n$.

Since, $\begin{bmatrix} \widetilde{Y}_1^n \\ \widetilde{Y}_2^n \end{bmatrix}$ has the same marginal as $\begin{bmatrix} Y_1^n \\ Y_2^n \end{bmatrix}$ given $\begin{bmatrix} X_1^n \\ X_2^n \end{bmatrix}$, we must have

$\Pr\Big\{g_1(\widetilde{Y}_1^n) \neq W_1 | W_1W_2 = w_1w_2\Big\} = P_{e,w_1w_2}^{(n)}(1)$ and $\Pr\Big\{\widetilde{Y}_1^n \neq Y_1^n | W_1W_2 = w_1w_2\Big\} = P_{e,w_1w_2}^{(n)}(2)$.

Therefore,

$$\begin{aligned} \Pr\Big\{G_2\left(Y_2^n\right) \neq W_1W_2 \Big| W_1W_2 = w_1w_2\Big\} &\leqslant \Pr\Big\{\widetilde{Y}_1^n \neq Y_1^n | W_1W_2 = w_1w_2\Big\} + \Pr\Big\{g_1(\widetilde{Y}_1^n) \neq W_1 | W_1W_2 = w_1w_2\Big\} \\ &\leqslant P_{e,w_1w_2}^{(n)}(1) + P_{e,w_1w_2}^{(n)}(2). \end{aligned}$$

Similarly $\Pr\Big\{G_2\left(Y_2^n\right) \neq W_1W_2 \Big| W_1W_2 = w_1w_2\Big\} \leqslant P_{e,w_1w_2}^{(n)}(1) + P_{e,w_1w_2}^{(n)}(2)$. Averaging over messages, we have a joint decoder whose $P_{e,\text{joint}}^{(n)}(k) \leqslant \lambda/2 + \lambda/2 = \lambda$. So $R \in \mathscr{C}_{\text{MAC}(2,2,2)}$. □

2 A Modified Interference Channel

Motivation: While in the above example, $a_{12} \geqslant 1$, $a_{21} \geqslant 1$ implies both decoders could decode both streams, this is not possible in general. Suppose each decoder decodes only a part of the other user's signal. Let us call this common. The undecoded signal is dedicated to the other user. The part that is decoded may enable a partial interference cancellation/mitigation. This motivates a splitting of the users data into a common and a dedicated portion.

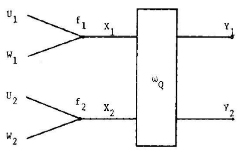

Figure 2: Modified Interference Channel.

Definition 6 (Code). *An (n, M_1, M_2, M_3, M_4) code for the channel $\left(\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}_1, \mathbb{Y}_2, p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)\right)$ consists of the following:*

1. *$[M_1] \times [M_2]$ is the index set of messages for user 1 and $[M_3] \times [M_4]$ that for user 2.*
2. *An encoder for user 1, $f_1 : [M_1] \times [M_2] \to \mathbb{X}_1^n$ and that for user 2 is $f_2 : [M_3] \times [M_4] \to \mathbb{X}_2^n$. The codebook can be represented by an ordered set*

$$c = \{f_1(\{1,1\}), f_1(\{1,2\}) \cdots, f_1(\{1,M_2\}), \cdots, f_1(\{M_1,M_2\}); f_2(\{1,1\}), \cdots, f_2(\{M_3,M_4\})\}.$$

3. *A decoding rule, $g_1 : \mathbb{Y}_1^n \to \phi \cup [M_1] \times [M_2] \times [M_4]$, and $g_2 : \mathbb{Y}_2^n \to \phi \cup [M_2] \times [M_3] \times [M_4]$.*

Definition 7 (Probability of error). *Let (W_1, W_2) be the message transmitted by user 1 and (W_3, W_4) by user 2. $P_{e,w_1w_2w_3w_4}^{(n)}(\ell)$, the conditional probability of error for output terminal ℓ when $(W_1W_2W_3W_4) = (w_1w_2w_3w_4)$ was transmitted is defined as before. and $P_e^{(n)}(\ell)$, the average probability of error for the code c (for output terminal ℓ) is also defined as before.*

Definition 8 (Achievability). *The rate quadruple* $(R_1, R_2, R_3 R_4)$ *is achievable, if for every* $\eta > 0$, $\lambda \in (0,1)$, *there exists a sequence of* (n, M_1, M_2, M_3, M_4) *codes that satisfy*

1. $P_e^{(n)}(\ell) \leq \lambda$, *and*
2. $\frac{\log_2 M_\ell}{n} > R_\ell - \eta$, $\ell \in [2]$,

for all sufficiently large n.