E2–301 Topics in Multiuser Communication September 18, 2007

Lecture 9 : Modified Interference Channel (I_m)

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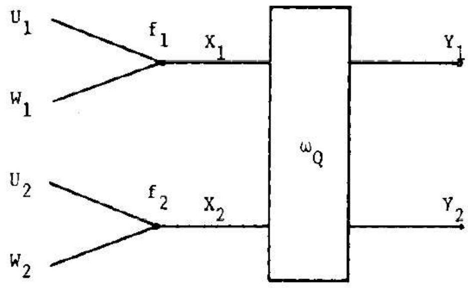

Figure 1: Modified Interference Channel.

Proposition 1. *If there is an (n, M_1, M_2, M_3, M_4) code on I_m with $P_e^{(n)}(k; I_m) \leqslant \lambda,\ k = 1, 2$ then there is an (n, M_1M_2, M_3M_4) code on I with corresponding $P_e^{(n)}(k; I) \leqslant \lambda,\ k = 1, 2.$*

Proof. Consider the mappings

$$y_1^n \quad \overset{g_1}{\longmapsto} \quad \widehat{w}_1\widehat{w}_2\widehat{w}_4 \quad \overset{h_1}{\longmapsto} \quad \widehat{w}_1\widehat{w}_2,$$

Define $G_1 := h_1(g_1(\cdot))$. Similarly define G_2. If G_k makes an error, so does g_k. So $P_e^{(n)}(k; I) \leqslant P_e^{(n)}(k; I_m) \leqslant \lambda, k = 1, 2.$ □

Corollary: If $(r_1, r_2, r_3, r_4) \in \mathscr{C}_{I_m}$, then $(r_1 + r_2, r_3 + r_4) \in \mathscr{C}_I$.

Definition 1 ($\mathscr{P}^*$).

$$\mathscr{P}^* \quad := \quad \left\{ Z = QU_{[4]}X_{[2]}Y_{[2]} \ \textit{such that}\ (1) - (3)\ \textit{below hold} \right\}.$$

(1) $p_{U_{[4]}|Q} = \prod_{j \in [4]} p_{U_j|Q}$.

(2) $X_1 = f_1(U_1U_2|Q)$, $X_2 = f_2(U_3U_4|Q)$, *deterministic functions.*

(3) $p_{QU_{[4]}X_{[2]}Y_{[2]}} = p_Q \cdot \left(\prod_{j \in [4]} p_{U_j|Q} \right) \left(p_{X_{[2]}|U_{[4]}Q} \right) \left(p_{Y_{[2]}|X_{[2]}} \right).$

Definition 2 ($\rho(S)$). *Let $\sum_1 := \{1, 2, 4\}$, and $\sum_2 := \{2, 3, 4\}$. Define $\rho_\ell : 2^{[4]} \to \mathbb{R}_+ \cup \{+\infty\}$ as*

$$\rho_\ell(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ I\left(U_S; Y_\ell | U_{\sum_\ell \setminus S}, Q\right) & \text{if } S \subseteq \sum_\ell \\ \infty & \text{if } S \nsubseteq \sum_\ell. \end{cases}$$

Define $\rho(S) := \min\{\rho_\ell(S), \ell \in [L]\}$.

Definition 3.

$$\mathscr{S}(Z) \quad := \quad \left\{ r \in \mathbb{R}_+^4 : r(S) \leqslant \rho(S),\ \forall S \subseteq [J] \right\}.$$

Definition 4.

$$\mathscr{S} := closure \left(\bigcup_{Z \in \mathscr{P}^*} \mathscr{S}(Z) \right).$$

Definition 5.

$$\mathscr{R}(Z) \quad := \quad \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : R_1 = r_1 + r_2, R_2 = r_3 + r_4 \text{ for some } r \in \mathscr{S}(Z) \right\}.$$

Definition 6.

$$\mathscr{R} := closure \left(\bigcup_{Z \in \mathscr{P}^*} \mathscr{R}(Z) \right).$$

Remark 1. *Both $\mathscr{R}$ and $\mathscr{S}$ are convex.*

Theorem 1. $\mathscr{S} \subseteq \mathscr{C}_{I_m}$ *and consequently* $\mathscr{R} \subseteq \mathscr{C}_I$.

Proof. Straight forward, via random coding, as in Ahlswede–Ulrey–Han generalisation, except that user 1 need not decode U_3 and user 2 U_1. Consequently, these conditions disappear and $\rho_1(S) = +\infty$ when $3 \in S$ and $\rho_2(S) = +\infty$ when $1 \in S$. □

1 A simplification:

Theorem 2. *For any $Z \in \mathscr{P}^*$, $\mathscr{R}(Z)$ is the polyhedron, $R_1 \leqslant \rho_1$, $R_2 \leqslant \rho_2$, $R_1 + R_2 \leqslant \rho_{12}$, $2R_1 + R_2 \leqslant \rho_{10}$, $R_1 + 2R_2 \leqslant \rho_{20}$, for ρ_1, ρ_2, ρ_{12}, ρ_{10}, ρ_{20} as in Eqns. (4.2) – (4.6) of HK 1981.*

Proof. **Fourier–Motzkin elimination**. Straight forward, but tedious. It gives bounds only of the form $R_1 \leqslant \cdots$, $R_2 \leqslant \cdots$, $R_1 + R_2 \leqslant \cdots$, $R_1 + 2R_2 \leqslant \cdots$, $2R_1 + R_2 \leqslant \cdots$. □

Remark 2. *Suppose (r_1, r_2, r_3, r_4) is a maximal extreme point of $\mathscr{S}(Z)$ ($R_{1e} = r_1 + r_2$, and $R_{2e} = r_3 + r_4$). A decrease of Δ in $r_1 + r_2$ may increase R_2 by at most 2Δ (r_3 may increase at most Δ. Similarly r_4. So $\Delta(r_3 + r_4) \leqslant 2\Delta$.) This implies $2R_1 + R_2 \leqslant 2R_{1e} + R_{2e}$ = constant. Similarly $R_1 + 2R_2 \leqslant$ constant.*