E2–301 Topics in Multiuser Communication	September 20, 2007
Lecture 10 : Fourier-Motzkin elimination, outer bounds of interference channels	
Instructor: Rajesh Sundaresan	*Scribe: Premkumar K.*

1 Fourier Motzkin elimination:

Solve $x_1, x_2, \cdots, x_n$ such that

$$\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i, \quad i = 1, 2, \cdots, m.$$

Pick a variable, say x_n. Eliminate it as follows. Assume $a_{in} \neq 0$.

$$a_{in}x_n \leqslant b_i - \sum_{j=1}^{n-1} a_{ij}x_j, \quad i = 1, 2, \cdots, m.$$

If $a_{in} > 0$, then upper bound $x_n \leqslant \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j = \beta_i$, otherwise lower bound $x_n \geqslant \alpha_i$. Eliminate x_n from all equations; replace by $\alpha_{i'} \leqslant \beta_i$ for every i', i such that i' yields a lower bound, i yields an upper bound. Let LB_n and UB_n be the set of indices that yield lower and upper bounds respectively. If this system has a solution in $n-1$ variables, then that solution with any x_n in $[\max_{i \in LB_n} \alpha_i \min_{i \in UB_n} \beta_i]$ is a solution to the original set.

2 Gaussian interference channel:

For the Gaussian interference channel defined earlier, $\mathscr{P}, \mathscr{P}^*$ depend on the power constraints P_1 and P_2. Note the following definitions.

- $\mathscr{P}^*(P_1, P_2) = \{Z \in \mathscr{P}^*, \mathrm{Var}(X_k) \leqslant P_k, k = 1, 2\}$
- $\mathscr{P}(P_1, P_2) = \{Z \in \mathscr{P}^*(P_1, P_2), \text{with } |\mathbb{Q}| = 1\}$
- $\mathscr{G} = \text{closure conv} \bigcup_{Z \in \mathscr{P}(P_1, P_2)} \mathscr{R}(Z)$
- $\mathscr{G}^* = \text{closure} \bigcup_{Z \in \mathscr{P}^*(P_1, P_2)} \mathscr{R}(Z)$
- $\mathscr{P}'(P_1, P_2) = \{Z \in \mathscr{P}(P_1, P_2) : U_1, U_2, U_3, U_4 \text{ are Gaussian }, U_1 + U_2 = X_1, U_3 + U_4 = X_2\}$
- $\mathscr{G}' = \text{closure conv} \bigcup_{Z \in \mathscr{P}'(P_1, P_2)} \mathscr{R}(Z).$
- Questions: Does correlation in $U_1U_2U_3U_4$ help? Is $\mathscr{G}' \subsetneq \mathscr{G}^*$? Is $\mathscr{G}^* \subsetneq \mathscr{G}'$?

3 Outer bounds:

(1) DMC.

Definition 1 (Π)**.**

$$\Pi := \{Z = QX_1X_2Y_1Y_2\widetilde{Y}_1\widetilde{Y}_2\widehat{Y}_1\widehat{Y}_2 \textit{ such that } (1) - (2) \textit{ below hold}\}.$$

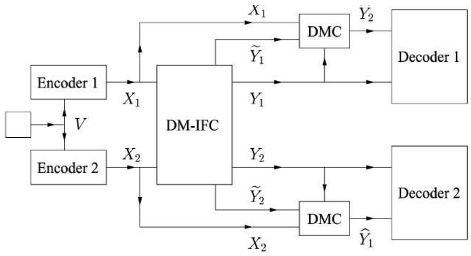

Figure 1: A statistical model for outer bound.

(1) $p_Z = p_Q \left(p_{X_1|Q} p_{X_2|Q}\right) \left(p_{Y_1Y_2|X_1X_2}\right) \left(p_{\widetilde{Y}_1\widetilde{Y}_2|X_1X_2Y_1Y_2}\right) \left(p_{\widehat{Y}_2|X_1Y_1\widetilde{Y}_1}\right) \left(p_{\widehat{Y}_1|X_2Y_2\widetilde{Y}_2}\right)$.

(2) $p_{\widehat{Y}_2|X_1X_2} = p_{Y_2|X_1X_2}$, $p_{\widehat{Y}_1|X_1X_2} = p_{Y_1|X_1X_2}$.

Definition 2.

$$\mathscr{R}_\Pi(Z) := \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \begin{array}{rcl} R_1 & \leqslant & I(X_1; Y_1|X_2Q) \\ R_2 & \leqslant & I(X_2; Y_2|X_1Q) \\ R_1 + R_2 & \leqslant & \min\left\{ I\left(X_1X_2; Y_1\widetilde{Y}_1|Q\right), I\left(X_1X_2; Y_2\widetilde{Y}_2|Q\right) \right\} \end{array} \right\}.$$

Definition 3.

$$\mathscr{R}_\Pi := \textit{closure} \bigcup_{Z\in\Pi} \mathscr{R}_\Pi(Z)$$

Theorem 1. $\mathscr{C}_I \subseteq \mathscr{R}_\Pi$

Proof. (1) Fix n, $p_Q(i) = \frac{1}{n}, p_{X_1X_2Y_1Y_2|Q}(x_1x_2y_1y_2|i) = p_{X_{1i}X_{2i}Y_{1i}Y_{2i}}$. As in the MAC's converse, $R_1 \leqslant I(X_1; Y_1|X_2Q)$ and $R_2 \leqslant I(X_2; Y_2|X_1Q)$.

(2)
- Now suppose the same codes are used in the new DMC with outputs $Y_1\widetilde{Y}_1$ at decoder 1 and $Y_2\widetilde{Y}_2$ at decoder 2.
- Decoder 1 gets $X_1^n, \widetilde{Y}_1^n, Y_1^n$; sends $\widetilde{Y}_1^n$ to DMC $p_{\widehat{Y}_2|X_1Y_1\widetilde{Y}_1}$ to get $\widehat{Y}_2^n$, applies decoder 2's decode function to get $\widehat{W}_2$ as reliably as decoder 2.
- Analogously, decoder 2 gets $\widehat{\widehat{W}}_2$ reliably, and moreover $\widehat{\widehat{W}}_1$ as reliably as decoder 1.
- Using the converse to Ahlswede–Ulrey–Han generalisation, since both can decode, we have a compound DMC that satisfies

$$\begin{array}{rcl} R_1 + R_2 & \leqslant & I(X_1X_2; Y_1\widetilde{Y}_1|Q) \\ R_1 + R_2 & \leqslant & I(X_1X_2; Y_2\widetilde{Y}_2|Q) \end{array}$$

□