E2–301 Topics in Multiuser Communication September 25, 2007

Lecture 11 : Gaussian interference channels, degraded interference channels, Z-channel

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A Recall:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{a_{12}} \\ \sqrt{a_{21}} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

A.1

$a_{12} \geqslant 1, a_{21} \geqslant 1 \implies$

$$\mathscr{C}_I = \left\{ \begin{array}{rcl} R_1 & \leqslant & \frac{1}{2}\log\left(1+P_1\right) \\ R_2 & \leqslant & \frac{1}{2}\log\left(1+P_2\right) \\ R_1+R_2 & \leqslant & \min\left\{\frac{1}{2}\log\left(1+P_1+a_{12}P_2\right), \frac{1}{2}\log\left(1+P_2+a_{21}P_1\right)\right\} \end{array} \right\}$$

A.2

$a_{12} \geqslant 1+P_1$ and $a_{21} \geqslant 1+P_2 \implies$ interference does not affect capacity region, $\mathscr{C}_I = [0,\ C(P_1)] \times [0,\ C(P_2)]$, where $C(P) := \frac{1}{2}\log(1+P)$.

B Degraded interference channel:

One receiver sees a statistically noisy version of the other's received signal.

B.1

- Since, $\mathscr{C}_I$ depends only on marginals, we may assume physical degradedness (more on this when we do broadcast channels).
- $y_1 = x_1 + \sqrt{a_{12}}x_2 + \xi_1$
  $y_2 = \sqrt{a_{21}}x_1 + x_2 + \xi_2$.
  If y_2 is a degraded version of y_1, we must have $\sqrt{a_{21}}x_1 + x_2 = c\cdot(x_1 + \sqrt{a_{12}}x_2)\ \forall x_1, x_2$, with $c \in (0,1]$. Thus $a_{21} = c$ and $a_{12}a_{21} = 1$.
- $a_{21} = 1$ is solved. So WMA $a_{21} \in (0,1)$.
- Equivalently, a degraded interference channel satisfies

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & c^{-1} \\ c & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad c \in (0,1).$$

C Z–channel:

$$\begin{aligned} y_1 &= x_1 + \xi_1 \\ y_2 &= \sqrt{a_{21}}x_1 + x_2 + \xi_2. \end{aligned}$$

We will denote its capacity region $\mathscr{C}_{I,Z}(a_{21})$.

C.1

Proposition 1. *(1) If $a_{21} \geqslant 1$, then*

$$\mathscr{C}_I = \left\{ \begin{array}{rcl} R_1 & \leqslant & C(P_1) \\ R_2 & \leqslant & C(P_2) \\ R_1 + R_2 & \leqslant & C(P_2 + a_{21}P_1) \end{array} \right\}.$$

(2) If $a_{21} \geqslant 1 + P_2$, then $\mathscr{C}_I = [0,\ C(P_1)] \times [0,\ C(P_2)]$.

Proof. (1)

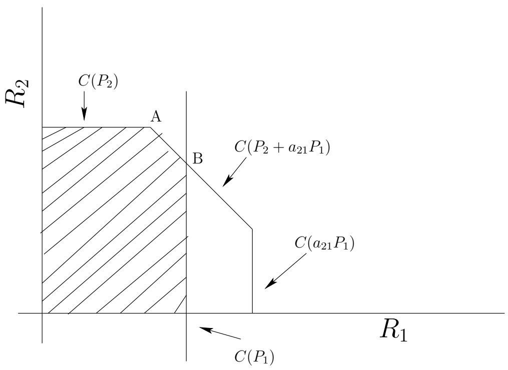

Figure 1: Achievable rate region when $1 \leqslant a_{21} \leqslant 1 + P_2$.

case 1: $C(P_1) < C(P_2 + a_{21}P_1) - C(P_2) \implies 1 + P_1 < \frac{1+P_2+a_{21}P_1}{1+P_2} = 1 + \frac{a_{21}P_1}{1+P_2} \implies a_{21} > 1 + P_2$.
Claim: $\mathscr{C}_I = [o, C(P_1)] \times [0, C(P_2)]$.

Proof. Strategy to achieve $(C(P_1), C(P_2))$:

- User 1 encodes as in a single user channel with rate $C(P_1)$. Decoder 1 can decode this signal.
- User 2 encodes in a similar fashion. Decoder 2 decodes signal 1 first. SNR $= \frac{a_{21}P_1}{P_2+1} > P_1$. $\implies$ decoding is reliable. Subtract, decode user 2's signal.

□

case 2: $1 \leqslant a_{21} \leqslant 1 + P_2$
Claim: $\mathscr{C}_I$ is the shaded region.

Proof. - A is achievable. User 1 encodes at $C(\frac{a_{21}P_1}{1+P_2})$ as in single user channel. User 2 encodes at $C(P_2)$.

- Decoder 1 sees SNR P_1, but rate is lower (corresponding SNR is $\frac{a_{21}P_1}{1+P_2}$). So decoder 1 decodes message 1. Decoder 2 employs successive interference cancellation.
- B is achievable. User 1 encodes at $C(P_1) = C(\frac{1}{a_{21}}(a_{21}P_1))$. User 2 splits powers as $a_{21} - 1$, $P_2 - (a_{21} - 1) \geqslant 0$ and encodes at rates $C(a_{21} - 1)$, $C(\frac{P_2-(a_{21}-1)}{1+(a_{21}-1)+a_{21}P_1})$. Clearly decoder 1 can decode at $C(P_1)$, its capacity. Decoder 2 decodes $C(\frac{P_2-(a_{21}-1)}{1+(a_{21}-1)+a_{21}P_1})$, then $C(P_1) = C(\frac{1}{a_{21}}(a_{21}P_1))$, then finally $C(a_{21} - 1)$. Sum rate is $C(1 + \frac{P_2-a_{21}}{a_{21}(1+P_1)}) + C(P_1) + C(a_{21} - 1) = C(P_2 + a_{21}P_1)$

□

(2) Obvious. Since $C(P_2 + a_{21}P_1) \geqslant C(P_1) + C(P_2)$.

□

C.2

Remark 1. *$\mathscr{C}_I$, when $a_{21} \in (0,1)$ is still open.*

C.3

Proposition 2. *The capacity region of the Z–channel with $a_{21} \in (0,1)$ equals the capacity region of the degraded interference channel with the same a_{21}.*

Proof. (**Transformation**)

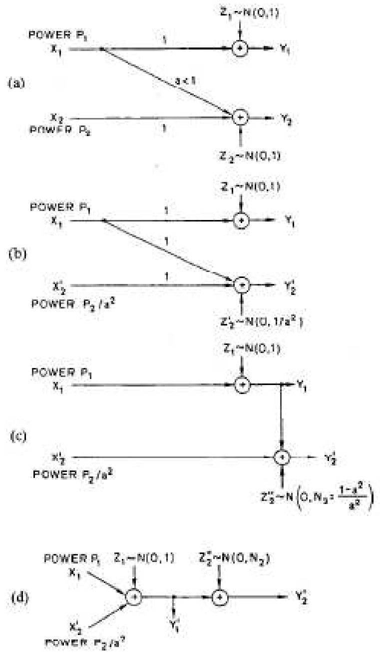

Figure 2: A statistical model for outer bound.

- $\mathscr{C}_I^{(a)} = \mathscr{C}_I^{(b)} = \mathscr{C}_I^{(c)}$ is clear. Also, $\mathscr{C}_I^{(d)} \subseteq \mathscr{C}_I^{(c)}$. This is because Y_2 has the same conditional output distribution in both cases. To decoder 1, provide x_2^n and $x_1^n + \xi_1^n$. Optimal use of x_2^n and $x_1^n + \xi_1^n$ by decoder 1 bounds $\mathscr{C}_I^{(d)}$. Clearly y_1^n can be discarded. Moreover, for decoding w_1, x_2^n provides no additional information and can be discarded. Optimal decoding then makes use of $x_1^n + \xi_1^n$ only, as in (c). So $\mathscr{C}_I^{(d)} \subseteq \mathscr{C}_I^{(c)}$.

- $\mathscr{C}_I^{(c)} \subseteq \mathscr{C}_I^{(d)}$: Take a code on (c). We will apply it on (d). User 2's decoder remains the same. Modify user 1's decoder to apply user 2's decoder first (after addition of noise), subtract user 2's signal, apply user 1's (c) decoder. We have thus simulated channel (c) on channel (d) with probability of decoder 2 error. so an achievable rate on (c) is achievable in (d).

□

D A bound on $\mathscr{C}_I$ for the degraded interference channel:

D.1

Proposition 3. *a) Let $a_{21} \in (0,1), a_{12} = 1/a_{21}$, so that y_2 is a degraded version of y_1. Then,*

$$\mathscr{C}_I \subseteq \left\{ \begin{array}{lll} R_1 & \leqslant & \frac{1}{2}\log\left(1 + t(P_1 + P_2/a_{21})\right) \\ R_2 & \leqslant & \frac{1}{2}\log\left(1 + (1-t)\frac{P_1+P_2/a_{21}}{1/a_{21}+t(P_1+P_2/a_{21})}\right) \end{array} \right\} =: \mathscr{C}_{BC}\left(P_1 + \frac{P_2}{a_{21}}\right)$$
$$\cap \ [0,\ C(P_1)] \times [0,\ C(P_2)]$$

b) Moreover, $\left(c(P_1), c\left(\frac{P_2}{1+a_{21}P_1}\right)\right)$ is achievable.

c) $\max\limits_{(R_1,R_2)\in\mathscr{C}_I} R_1 + R_2 = C(P_1) + C\left(\frac{P_2}{1+a_{21}P_1}\right)$.

Proof. a) Pool resources and consider the broadcast channel.

$$\begin{aligned} y_1 &= x_1 + \frac{1}{\sqrt{a_{21}}}x_2 + \xi_1 \\ y_2' = \frac{1}{\sqrt{a_{21}}}y_2 &= x_1 + \frac{1}{\sqrt{a_{21}}}x_2 + \frac{\xi_2}{\sqrt{a_{21}}}. \end{aligned}$$

$\mathscr{C}_I \subseteq \mathscr{C}_{\text{BC}}\left(P_1 + \frac{P_2}{a_{21}}\right)$ which is given by the first set in the upper bound (refer sec. E.2). Futhermore, $\mathscr{C}_I \subseteq [0,\ C(P_1)] \times [0,\ C(P_2)]$. This yields (a).

b) Choose t so that $t(P_1+P_2/a_{21}) = P_1$. Then $(1-t)(P_1+P_2/a_{21}) = \frac{P_2}{a_{21}}$. Since superposition coding in the BC disregards the stronger signal knowledge, and power constraints at the two terminals are met, the encoding can happen at separate places with superposition done by the channel. So $\left(c(P_1), c\left(\frac{P_2}{1+a_{21}P_1}\right)\right)$ is achievable.

c)

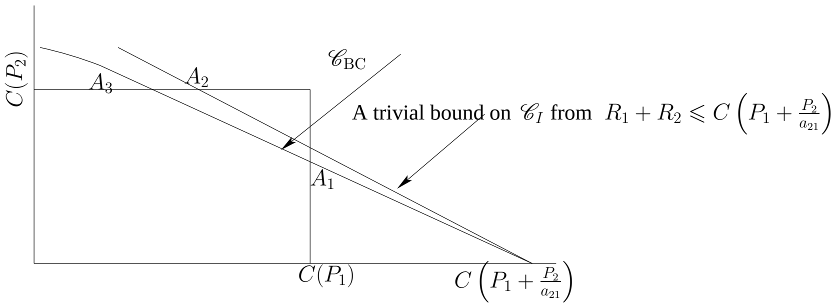

Figure 3: Outer bound of $\mathscr{C}_I$.

$\mathscr{C}_{\text{BC}}$ boundary parametrized by t is concave function of R_1. |Slope| $\leqslant 1$, negative. $\implies \max R_1 + R_2 \leqslant$ max value of A_1. But A_1 is achievable.

□

E Putting our results together:

E.1

Proposition 4. *$\mathscr{C}_I$ for the Z–channel with $a_{21} \in (0,1)$ satisfies*

- $\mathscr{C}_I \subseteq \mathscr{C}_{BC}(P_1 + P_2/a_{21}) \cap [0,\ C(P_1)] \times [0,\ C(P_2)]$
- $\left(C(P_1), C\left(\frac{P_2}{1+a_{21}P_1}\right)\right)$ *is achievable.*
- $\max\limits_{(R_1,R_2)\in\mathscr{C}_I} R_1 + R_2 = C(P_1) + C\left(\frac{P_2}{1+a_{21}P_1}\right).$

Proof. Follows from the equivalence of degraded interference channel and Z–channel when $a_{21} \in (0,1)$.

□

E.2

Proposition 5. *Consider the interference channel with $a_{21} \in (0,1)$ and a_{12} arbitrary.*

(1) $\mathscr{C}_I \subseteq \mathscr{C}_{BC}(P_1 + P_2/a_{21}) \cap [0,\ C(P_1)] \times [0,\ C(P_2)]$

(2) If $a_{12} \in (0,1)$, then $\mathscr{C}_I \subseteq \mathscr{C}_{BC}(P_1 + P_2/a_{21}) \cap \mathscr{C}_{BC}(P_2 + P_1/a_{12}) \cap [0,\ C(P_1)] \times [0,\ C(P_2)]$.

Proof. Sufficient to prove (1). A genie provides x_2^n and therefore $x_1^n + \xi_1^n$ to decoder 1. Optimal processing will discard x_2^n, y_1^n and operate on $x_1^n + \xi_1^n$, the output of the Z–channel. So $\mathscr{C}_I \subseteq \mathscr{C}_{I,Z}(a_{21})$. (1) then follows from Prop. 4.

□

F Other results:

- I.Sason's achievable rate region: easier to compute subset of $\mathscr{G}^*$(HK region).
- I.Sason's shows that if $a_{12} = a_{21} \in (0,1)$, for high enough SNR, TDM/FDM is not optimal.
- Telatar and Tse : A new expression for converse with difference between $\mathscr{C}_{\text{HK}}$ and $\mathscr{C}_{\text{outer}}$ within 1 bit per received dimension (scalar case 1 bit/sample use).