E2–301 Topics in Multiuser Communication	October 09, 2007

Lecture 13 : Broadcast channels – Outer bounds

Instructor: Rajesh Sundaresan *Scribe: Premkumar K.*

Theorem 1. *(Marton 1979)* $\mathscr{R} \subseteq \mathscr{C}_{BC}$.

Proof. By symmetry, we will prove (R_1, R_2) is achievable, where

$$\begin{aligned} R_1 &= I(U_0U_1;Y_1) = I(U_0;Y_1) + I(U_1;Y_1|U_0) \\ R_2 &= \min\left\{I(U_0;Y_1), I(U_0;Y_2)\right\} + I(U_1;Y_1|U_0) + I(U_2;Y_2|U_0) - I(U_1;U_2|U_0) - I(U_0;Y_1) - I(U_1;Y_1|U_0) \\ &= I(U_2;Y_2|U_0) - I(U_1;U_2|U_0) - \left|I(U_0;Y_1) - I(U_0;Y_2)\right|_+ \end{aligned}$$

If $R_2 = 0$, $R_1 = I(U_0U_1;Y_1)$ is achievable as will be clear from below. Let us analyze the case when $R_2 > 0$. Fix $\lambda, \delta, \eta > 0$. Fix n.

$$\begin{aligned} I &= \left\lceil 2^{n(I(U_0;Y_1)-\eta)} \right\rceil \\ J &= \left\lceil 2^{n(I(U_1;Y_1|U_0)-\eta)} \right\rceil \\ K &= \left\lceil 2^{n\left(I(U_2;Y_2|U_0)-I(U_1;U_2|U_0)-\left|I(U_0;Y_1)-I(U_0;Y_2)\right|_+-2\eta\right)} \right\rceil \\ \text{and strangely,} \quad L &= \left\lceil 2^{n(I(U_1;U_2|U_0)+\eta)} \right\rceil \end{aligned}$$

Code:

- $u_0^n(i), i \in [I], \sim P_{U_0^n} = \prod_{i=1}^{n} P_{U_0}$
- $u_1^n(ij), i \in [I], j \in [J], \sim P_{U_1^n|U_0^n}(\cdot|u_0^n(i))$
- $u_2^n(ik\ell), i \in [I], k \in [K], \ell \in [L], \sim P_{U_2^n|U_0^n}(\cdot|u_0^n(i))$

Encoder:

- Message (ij) from source 1 and k from source 2. Identify $u_0^n(i)u_1^n(ij)$.
- Next, in the set $u_2^n(ik\cdot)$, search for an ℓ such that $u_0^n(i)u_1^n(ij)u_2^n(ik\ell) \in T_{\frac{\delta}{2}}^{(n)}(U_0U_1U_2)$. The ℓ is denoted as $\ell(ijk)$.
- Pick x^n such that $u_0^n(i)u_1^n(ij)u_2^n(ik\ell)x^n(ijk) \in T_{\delta}^{(n)}(U_0U_1U_2X)$.

Decoder:

1) Look for the unique ij such that $u_0^n(i)u_1^n(ij)y_1^n \in T_{2\delta}^{(n)}(U_0U_1Y_1)$
2) Look for the unique ik such that $u_0^n(i)u_2^n(ik\ell)y_2^n \in T_{2\delta}^{(n)}(U_0U_2Y_2)$ for some ℓ.

Analysis:

0) WMA $ijk = 111$ by symmetry.
1) $u_0^n(1)u_1^n(11) \in T_{\frac{\delta}{4}}^{(n)}(U_0U_1) w.p. \geqslant 1-\delta$.
2) Encoding error:

$\nexists \ell : u_0^n(1)u_1^n(11)u_2^n(11\ell) \in T_{\frac{\delta}{2}}^{(n)}(U_0U_1U_2)$. If we pick enough L, we should get a hit.

$$\begin{aligned}
& Pr\Big\{\bigcap_{\ell=1}^{L}\{u_0^n(1)u_1^n(11)u_2^n(11\ell) \notin T_{\frac{\delta}{2}}^{(n)}(U_0U_1U_2)\}\big|u_0^n(1)u_1^n(11)\Big\} \\
= & \prod_{\ell=1}^{L} Pr\Big\{u_0^n(1)u_1^n(11)u_2^n(11\ell) \notin T_{\frac{\delta}{2}}^{(n)}(U_0U_1U_2)\big|u_0^n(1)u_1^n(11)\Big\} \\
\leqslant & \left(1-2^{-n(I(U_1;U_2|U_0)-7\delta)}\right)^L \\
\leqslant & e^{-L2^{-n(I(U_1;U_2|U_0)-7\delta)}} \\
\leqslant & e^{-2^{-n(\eta-7\delta)}} \downarrow 0, \quad \text{if } \eta > 7\delta
\end{aligned}$$

3) Decoder 1's failure:

- There is another $(ij) \neq (11)$.
- If $u_0^n(i)u_1^n(ij)y_1^n \in T_{2\delta}^{(n)}(U_0U_1Y_1)$, then $u_0^n(i)y_1^n \in T_{2\delta}^{(n)}(U_0Y_1)$.

$$\begin{aligned}
prob. & \leqslant (I-1)2^{-nI(U_0;Y_1)+7n\delta} \\
& \leqslant 2^{-n(\eta-7\delta)}.
\end{aligned}$$

Decoder 2's failure:

- There is another $(ik) \neq (11)$ such that $u_0^n(i)u_2^n(ik\ell)y_2^n \in T_{2\delta}^{(n)}(U_0U_1Y_2)$ for some $\ell \in [L]$.
- If $i \neq 1$, $\underbrace{u_0^n(i)u_2^n(ik\ell)y_2^n}_{\text{independent of } Y_2} \in T_{2\delta}^{(n)}(U_0U_2Y_2)$. So,

$$\begin{aligned}
prob. & \leqslant (I-1)KL2^{-nI(U_0U_2;Y_2)+7n\delta} \\
& \leqslant 2^{-n(I(U_0;Y_1)+I(U_2;Y_2|U_0)-|I(U_0;Y_1)-I(U_0;Y_2)|_+-eta-7\delta)} \\
& \leqslant 2^{-n(\eta-7\delta)}.
\end{aligned}$$

  Hence, $I(U_0;Y_1)+I(U_2;Y_2|U_0)-|I(U_0;Y_1)-I(U_0;Y_2)|_+ \leqslant I(U_0U_2;Y_2)$.
- If $i = 1$, $u_0^n(1)u_2^n(1k\ell)y_2^n \in T_{2\delta}^{(n)}(U_0U_2Y_2)$, for some $k > 1$ and ℓ.

$$\begin{aligned}
prob. & \leqslant (K-1)L2^{-nI(U_2;Y_2|U_0)+7n\delta} \\
& \leqslant 2^{nI(U_2;Y_2|U_0)-n\eta/2-nI(U_2;Y_2|U_0)+n7\delta} \\
& \leqslant 2^{-n(\eta/2-7\delta)}.
\end{aligned}$$

  Take $\delta < \eta/14$ and n sufficiently large enough.

□