E2–301 Topics in Multiuser Communication	October 11, 2007
Lecture 14 : Broadcast channels – Converse	
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Theorem 1. *(Marton 1979)* $\mathscr{R} \subseteq \mathscr{C}_{BC}$.

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Theorem 2. 1. *(körner–Marton) Let*

$$\mathscr{R}_{out,1} = \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \begin{array}{rcl} R_1 & \leqslant & I(X;Y_1) \\ R_2 & \leqslant & I(U;Y_2) \\ R_1 + R_2 & \leqslant & I(U;Y_2) + I(X;Y_1|U) \\ & & \text{for some } U\phi\phi XY_{[2]} \in \mathscr{P}, |\mathbb{U}| \leqslant |\mathbb{X}| + 2 \end{array} \right\}.$$

Then $\mathscr{C}_{BC} \subseteq \mathscr{R}_{out,1}$.

2. *(ElGamal) Let* $\mathscr{R}_{out,2}$ *be the subset of* $\mathscr{R}_{out,1}$ *with* $R_1 \leqslant I(X;Y_1)$ *replaced by* $R_1 + R_2 \leqslant I(X;Y_1)$. *If the channel has a more capable component then* $\mathscr{C}_{BC} \mathrel{\mathring{=}} \mathscr{R}_{out,2}$.

Remark 1. *Both* $\mathscr{R}_{out,1}, \mathscr{R}_{out,2}$ *are convex.*

Proof. By Fano's inequality:

$$\begin{aligned} nR_k &= H(W_k) = H(W_k|Y_k^n) + I(W_k;Y_k^n) \\ &\leqslant I(W_k;Y_k^n) + nR_k P_e^{(n)}(k) + 1 \\ &\leqslant I(W_k;Y_k^n) + n\epsilon_n(k) \end{aligned}$$

From the Markov chain and memoryless property, $U_i := W_2 Y_2^{i+1\to n} Y_1^{1\to i-1}$, we have

a)

$$I(W_1;Y_1^n) \leqslant I(X^n;Y_1^n) \leqslant \sum_{i=1}^n I(X_i;Y_{1i}) = \sum_{i=1}^n I(U_i;Y_{2i})$$

b)

$$\begin{aligned} I(W_2;Y_2^n) &= \sum_{i=1}^n I(W_2;Y_{2i}|Y_2^{i+1\to n}) \\ &\leqslant \sum_{i=1}^n I(W_2 Y_2^{i+1\to n};Y_{2i}) \\ &\leqslant \sum_{i=1}^n I(W_2 Y_2^{i+1\to n};Y_1^{1\to i-1}) \end{aligned}$$

c)

$$\begin{aligned} I(W_1;Y_1^n) + I(W_2;Y_2^n) &\leqslant I(W_1;W_2Y_1^n) + I(W_2;Y_2^n) \\ &= I(W_1;Y_1^n|W_2) + I(W_2;Y_2^n) \quad W_2 \text{is indep. of } W_1 \\ &= \sum_{i=1}^n I(W_1;Y_{1i}|W_2Y_1^{1\to i-1}) + I(W_2;Y_{2i}|Y_2^{i+1\to n}) \\ &\leqslant \sum_{i=1}^n I(W_1Y_2^{i+1\to n};Y_{1i}|W_2Y_1^{1\to i-1}) + I(W_2;Y_{2i}|Y_2^{i+1\to n}) \end{aligned}$$

Note:

(a) $I(W_1 Y_2^{i+1\to n}; Y_{1i}|W_2 Y_1^{1\to i-1}) = I(Y_2^{i+1\to n}; Y_{1i}|W_2 Y_1^{1\to i-1}) + I(W_1; Y_{1i}|W_2 Y_1^{1\to i-1} Y_2^{i+1\to n})$

(b) $I(W_2 Y_1^{1\to i-1} Y_2^{i+1\to n}; Y_{2i}) = I(Y_2^{i+1\to n}; Y_{2i}) + I(W_2; Y_{2i}|Y_2^{i+1\to n}) + I(Y_1^{1\to i-1}; Y_{2i}|W_2 Y_2^{i+1\to n})$.

(c)

$$\begin{aligned}
\sum_{i=1}^{n} I(Y_2^{i+1\to n}; Y_{1i}|W_2 Y_1^{1\to i-1}) &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} I(Y_{2j}; Y_{1i}|W_2 Y_1^{1\to i-1} Y_2^{j+1\to n}) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{j-1} I(Y_{2j}; Y_{1i}|W_2 Y_1^{1\to i-1} Y_2^{j+1\to n}) \\
&= \sum_{j=1}^{n} I(Y_{2j}; Y_1^{1\to j-1}|W_2 Y_2^{j+1\to n}) \\
&= \sum_{i=1}^{n} I(Y_{2i}; Y_1^{1\to i-1}|W_2 Y_2^{i+1\to n})
\end{aligned}$$

So

$$\begin{aligned}
I(W_1; Y_1^n) + I(W_2; Y_2^n) &\leqslant \sum_{i=1}^{n} I(Y_2^{i+1\to n}; Y_{1i}|W_2 Y_1^{1\to i-1}) + I(W_1; Y_{1i}|W_2 Y_1^{1\to i-1} Y_2^{i+1\to n}) \\
&\quad + I(W_2; Y_1^{1\to i-1} Y_2^{i+1\to n}; Y_{2i}) - I(Y_1^{1\to i-1}; Y_{2i}|W_2 Y_2^{i+1\to n}) - I(Y_2^{i+1\to n}; Y_{2i}) \\
&\leqslant \sum_{i=1}^{n} I(W_1; Y_{1i}|U_i) + I(U_i; Y_{2i}).
\end{aligned}$$

$W_1 U_i \to X_i \to Y_{[2]i}$ is a Markov chain.
$(W_{[2]} Y_1^{1\to i-1} Y_2^{i+1\to n}) \to X_i \to Y_{[2]i}$
$\implies I(W_1; Y_{1i}|U_i) = I(WU_i; Y_{1i}) - I(U_i; Y_{1i}) \leqslant I(X_i; Y_{1i}) \leqslant I(X_i U_i; Y_{1i}) - I(U_i; Y_{1i}) = I(X_i; Y_{1i}; U_i)$

d) By Symmetry,

$$\leqslant \sum_{i=1}^{n} I(W_2; Y_{2i}|U_i') + I(U_i'; Y_{1i})$$

and $(W_2 U_i') \to X_i \to (Y_{1i} Y_{2i})$ is a Markov chain and therefore

$$\begin{aligned}
&\leqslant \sum_{i=1}^{n} I(X_i; Y_{2i}|U_i') + I(U_i'; Y_{1i}) \\
\text{More capable} \implies I(X_i; Y_{2i}|U_i') &\leqslant I(X_i; Y_{1i}|U_i') \\
\implies R_1 + R_2 &\leqslant \sum_{i=1}^{n} I(X_i; Y_{1i}), \quad \text{by the M.C. property.} \\
&= \sum_{i=1}^{n} I(X_i; Y_{1i}), \quad \text{by the M.C. property.}
\end{aligned}$$

e) Now let $U = (I, U_I)$, I uniform in $[n]$.

$$\begin{aligned}
X\big|U &:= X_I. \\
Y_{[2]}\big|U &:= Y_{[2]I}. \\
p_{UXY_{[2]}} &= p_U \cdot p_{X|U} \cdot \underbrace{p_{Y_{[2]}|XU}}_{p_{Y_{[2]I}|X_I I U_I} = p_{Y_{[2]}|X}(y_{[2]I}|x_I)}
\end{aligned}$$

So $U \to X \to Y_{[2]}$ is a Markov chain.

From (a), (b), (c) and (d), (1) follows. Using (d), (2) follows. The cardinality bounds follow from the constraints

1. $p_X(x) = \text{const}, \; x \in \mathbb{X}$ ($|\mathbb{X}| - 1$ constraints)
2. $I(X; Y_1|U) = \text{const.}$
3. $I(U; Y_2) = \text{const.} = H(Y_2) - H(Y_2|U)$

Since there are $|\mathbb{X}| + 1$ constraints, $|\mathbb{U}| \leqslant \mathbb{X} + 2$ by Caratheodory's theorem.

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