E2–301 Topics in Multiuser Communication	October 16, 2007
Lecture 15 : Connections between Superposition and Marton's Theorem	
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A Connections between superposition and Marton's Theorem

- We may consider, as did Han – Kobayashi in their interference channel paper, superposition. This was Cover's technique (1975), the basis for HK's result.
- Consider U_0, U_1, U_2 independent. U_0 decoded by both. U_1 and U_2 by users 1 and 2 respectively.

$$\begin{aligned} \mathscr{P}_1 &:= \left\{Z = U_{[|3|]} X Y_{[2]}\right\} \\ \text{and } \mathscr{P}_1^* &:= \left\{Z = Q U_{[|3|]} X Y_{[2]}, \quad \text{with } U_0U_1U_2 \text{ consitionally independent given } Q\right\} \end{aligned}$$

- Define

$$\mathscr{S}(Z) = \left\{ (r_1, r_2, r_0) \left| \begin{array}{rcl} r_0 & \leqslant & I(U_0; Y_1|U_1Q) \\ r_1 & \leqslant & I(U_1; Y_1|U_0Q) \\ r_0 + r_1 & \leqslant & I(U_0U_1; Y_1|Q) \\ r_0 & \leqslant & I(U_0; Y_2|U_2Q) \\ r_2 & \leqslant & I(U_2; Y_2|U_0Q) \\ r_0 + r_2 & \leqslant & I(U_0U_2; Y_2|Q) \end{array} \right. \right\}$$

- Intersection of two polymatroids in 2 dimensions, extended to three dimensions in different directions. $\mathscr{S}^* := \text{cl} \bigcup_{Z \in \mathscr{P}_1^*} \mathscr{S}(Z)$ is achievable with common information.
- (Cover, 1975) Define $\mathscr{S} := \text{cl. conv} \bigcup_{Z \in \mathscr{P}_1} \mathscr{S}(Z) = \text{ cl. conv. } \mathscr{S}_0$.

  Moreover

$$\begin{aligned} \mathscr{D} &:= \text{closure } \left\{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}\right\} \\ \mathscr{D}_0 &:= \text{closure conv } \left\{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}_0\right\} \\ \mathscr{D}^* &:= \text{closure } \left\{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}^*\right\} \\ \mathscr{D}_0^* &:= \text{closure } \left\{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}_0^*\right\} \end{aligned}$$

- Hajek–Pursley (1979) consider $\mathscr{D}_0$ and show that

$$\mathscr{D}_0 = \text{closure conv} \left\{ (R_1, R_2) \left| \begin{array}{rcl} R_1 & \leqslant & I(U_0U_1; Y_1) \\ R_2 & \leqslant & I(U_0U_2; Y_2) \\ R_1 + R_2 & \leqslant & \min\{I(U_0; Y_k), k = 1, 2\} + I(U_1; Y_1|U_0) + I(U_2; Y_2|U_0) \\ & & \text{such that 1)} U_{[|3|]} \to X \to Y_{[2]}, \\ & & \text{2)} U_{[|3|]} \text{ are independent.} \end{array} \right. \right\}$$

- Can easily extend equivalence of $\mathscr{D}^*$ with

$$\text{closure} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \begin{array}{rcl} R_1 & \leqslant & I(U_0U_1; Y_1|Q) \\ R_2 & \leqslant & I(U_0U_2; Y_2|Q) \\ R_1 + R_2 & \leqslant & \min\{I(U_0; Y_k|Q), k = 1, 2\} + I(U_1; Y_1|U_0Q) + I(U_2; Y_2|U_0Q) \\ & & \text{for some } QU_{[|3|]}XY_{[2]} \text{ that satisfies} \\ & & \text{1) Given } Q, U_0U_1U_2 \text{ are mutually independent,} \\ & & \text{2)} QU_0U_1U_2 \to X \to Y_{[2]} \end{array} \right\}$$

- With $Q\bar{U}_0$, it is easy to see that this region belongs to Marton's region.
- Inclusion in the other direction, or otherwise is not known, i.e., given Q, $U_0U_1U_2$ may lead to a larger region. Marton shows $\mathscr{R} \supsetneq \mathscr{D}_0$
- Is$\mathscr{D}_0 = \mathscr{D}$?
  $\mathscr{S}_0 = \bigcup\limits_{Z \in \mathscr{P}_1} \mathscr{S}(Z)$, $\mathscr{S} =$ closure conv $\mathscr{S}_0$.
- Clearly, since $\mathscr{S}_0 \subseteq \mathscr{S}$, we have $\mathscr{D}_0 \subseteq \mathscr{D}$.
- Let $(R_1, R_2) \in \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}\}$.
  $\implies \exists s_1 \geqslant 0, s_2 \geqslant 0$ with $(r_1, r_2, s_1 + s_2) \in \mathscr{S}$. If $(r_1, r_2, s_1 + s_2) \in \mathscr{S}_0$, then $(R_1, R_2) \in \mathscr{D}_0$ and we are done. If $(r_1, r_2, s_1 + s_2)$ is a limit point of conv $\mathscr{S}_0$, then an ϵ ball around this contains an $(r_1', r_2', s_1' + s_2') \in$ conv $\mathscr{S}_0$, $(r_1', r_2', s_1' + s_2') \neq (r_1, r_2, s_1 + s_2)$, i.e.,

$$(r_1', r_2', s_1' + s_2') = \sum_{\ell=1}^{L} \lambda_\ell \underbrace{(r_1'(\ell), r_2'(\ell), s_1'(\ell) + s_2'(\ell))}_{\in \mathscr{S}_0}.$$

- If we split $s_1' + s_2'$ into s_1', s_2' close enough to s_1, s_2, yet positive, and furthermore split s_1', s_2' into $s_1'(\ell), s_2'(\ell)$, positive, we get

$$(r_1' + s_1', r_2' + s_2') = \underbrace{\sum_{\ell=1}^{L} \lambda_\ell \underbrace{(r_1'(\ell) + s_1'(\ell), r_2'(\ell) + s_2'(\ell))}_{\in \{(r_1+s_1, r_2+s_2) : (r_1, r_2, s_1+s_2) \in \mathscr{S}_0\}}}_{\in \text{ conv } \{(r_1+s_1, r_2+s_2) : (r_1, r_2, s_1+s_2) \in \mathscr{S}_0\}}.$$

  If $(r_1' + s_1', r_2' + s_2') \neq (r_1 + s_1, r_2 + s_2)$, we will have shown $(r_1 + s_1, r_2 + s_2)$ is a limit point of the conv. $\{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathscr{S}_0\}$, and hence in $\mathscr{D}_0$.
- Note that once a suitable s_1', s_2' is picked, since $s_1' + s_2'$ is fixed and equals $\sum\limits_{\ell=1}^{L} \lambda_\ell (s_1'(\ell) + s_2'(\ell))$, the individual $s_1'(\ell), s_2'(\ell)$ may be picked arbitrarily ($s_1'(\ell) \geqslant 0, s_2'(\ell) \geqslant 0$, $s_1'(\ell) + s_2'(\ell) =$ a given value).
- If $s_1 = s_2 = 0$.
  - Suppose $s_1' + s_2' = 0$. Then $s_1' = s_2' = 0$ and since $(r_1', r_2', 0) \neq (r_1, r_2, 0)$, we must have $(r_1 + s_1, r_2 + s_2) \neq (r_1' + s_1', r_2' + s_2')$.
  - Suppose $s_1' + s_2' > 0$. If $r_1 \neq r_1'$ choose $s_1' = 0$; else choose $s_2' = 0$. Then $(r_1 + s_1, r_2 + s_2) \neq (r_1' + s_1', r_2' + s_2')$.
- If $s_1 > 0, s_2 > 0$. Take ϵ sufficiently small so that $s_1 - \epsilon > 0$, $s_2 - \epsilon > 0$.
  - If $r_1 \neq r_1'$, choose $s_1' = s_1$; else choose $s_2' = s_2$. (In this case $r_1 = r_1'$. If $s_1' = s_2$, then we must have $r_2' = r_2$). This ensures $(r_1' + s_1', r_2' + s_2') \neq (r_1 + s_1, r_2 + s_2)$.
- If $s_1 = 0, s_2 > 0$.
  - If $r_1' \neq r_1$, choose $s_1' = 0$.
  - If $r_1' = r_1$.
    Suppose $r_2' - r_2 \neq s_1 + s_2 - s_1' - s_2'$. Then set $s_1' = 0$. Then too $r_2' + s_2' \neq r_2 + s_2$.

– If $r_1' = r_1$ and $r_2' - r_2 = s_2 - s_1' - s_2'$ (under $s_1 = 0, s_2 > 0$). Choose $s_1' = \frac{|s_1+s_2-(s_1'+s_2')|}{2}$ $(\leqslant \epsilon)$. ϵ small enough so that $\epsilon < 2s_2/3$.

Then set $s_2' + \frac{|s_1+s_2-(s_1'+s_2')|}{2} = s_2' + s_1' \geqslant s_2 + s_1' - \epsilon$
$\implies s_2' \geqslant 0, |s_2 - s_2'| \leqslant 3\epsilon/2$.

Thus $d\Big((r_1 + s_1, r_2 + s_2), (r_1' + s_1', r_2' + s_2')\Big) \leqslant \sqrt{\frac{9\epsilon^2}{4} \times 2} = \frac{3}{\sqrt{2}}\epsilon$ (and $(r_1 + s_1, r_2 + s_2) \neq (r_1' + s_1', r_2' + s_2')$). Thus $\mathscr{D} = \mathscr{D}_0$.