Lecture 19 : Network–Cutset Bounds

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1 Network preliminaries

Definition 1 (Directed Graph). *$(N, \mathbb{B}, c)$ is a directed graph consisting of a set of nodes N, the directed edge set $\mathbb{B} \subset N \times N$ (an edge is an ordered pair $ij, i, j \in N$) and the capacity function $c : \mathbb{B} \to \mathbb{R}_+$.*

Definition 2 (Path). *Path of length k from $i \in N$ to $j \in N$ is the sequence of nodes $P : i = i_0, i_1, i_2, \cdots, i_{k-1}, i_k = j$ such that*

- *$i_1, i_2, \cdots, i_{k-1}$ are distinct, different from i and j*
- *$i_m i_{m+1} \in \mathbb{B}$.*

Definition 3 (Cycle). *A path with $i_0 = i_k$.*

Definition 4 (Cut). *$\big(\mathbb{M}, N\backslash\mathbb{M}\big)$ where $\mathbb{M} \subseteq N$.*

Definition 5 ($c(\mathbb{M}, \mathbb{L})$). *Suppose $\mathbb{M}, \mathbb{L} \subseteq N$. Define*

$$c(\mathbb{M}, \mathbb{L}) \quad := \sum_{ij \in \mathbb{M}\mathbb{L} \cap \mathbb{B}} c_{ij}$$

Definition 6 (net $c(i)$).

$$\begin{aligned} \textit{net} \;\; c(i) &:= c(\{i\}, N) - c(N, \{i\}) \\ &= \textit{outflow including self loop } (i,i) - \textit{inflow including self loop } (i,i) \\ &= \textit{net outflow from } i. \end{aligned}$$

•$[J] \subseteq N$ represents supply nodes; the first J nodes.
•$t_0 \in N\backslash[J]$ represents sink node.
•$N\backslash\big([J] \cup \{t_0\}\big)$ intermediate nodes.
Assumptions

1) For each $j \in [J]$, there is atleast one path from j to t_0.
2) No cycles.

Definition 7 (Feasible flow). *$f : \mathbb{B} \to \mathbb{R}_+$ such that*

1) $\forall ij \in \mathbb{B}$, $0 \leqslant f_{ij} \leqslant c_{ij}$

2)

$$\begin{aligned} \textit{net}(f,k) &\geqslant 0, && \textit{if } k \in [J] \\ \textit{net}(f,k) &\leqslant 0, && \textit{if } k = t_0 \\ \textit{net}(f,k) &= 0, && \textit{otherwise} \end{aligned}$$

Lemma 1. *(***Meggido, 1974***) Let ρ be the min–cut capacity function defined on subsets of $[J]$ as follows.*

$$\rho(S) \quad := \quad \min\big\{c(\mathbb{M}, N - \mathbb{M}) : S \subseteq \mathbb{M}, t_0 \in N - \mathbb{M}, M \subseteq N\big\}.$$

- *Then ρ is a sub–modular rank function.*

- *If* $R(S) \leqslant \rho(S), \forall S \subseteq \Sigma_0$, *then there is a flow such that* $R_s = net(f, S), \forall S \in [J]$. *If there exists a flow such that* $R_s = net(f, S), \forall S \in [J]$, *then* $R(S) \leqslant \rho(S), \forall S \subseteq [J]$.

Remark 1. *Generalization of max flow min cut theorem for multiple source nodes, single destination.*

Remark 2. *Any flow is a sum of path flows. In a path* P_{sk}, *flow is* R_{sk} *on all edges.*

Lemma 2. *(***Ford–Fulkerson***) For any flow* f, *there exist paths* P_{sk} $(s \in [J],\ k \in [m_s])$, *from* s *to* t_0 *and associated nonnegative real numbers* R_{sk} $(s \in [J],\ k \in [m_s])$ *such that*

1) $net\ (f, s) = \sum_{k=1}^{m_s} R_{sk},\ \forall s \in [J]$

2) $\sum\limits_{P_{sk}: ij \in P_{sk}} R_{sk} \leqslant c_{ij},\ \forall ij \in \mathbb{B}$