Lecture 20 : Multi–terminal Source Coding over a Network to a Single Destination

Instructor: Rajesh Sundaresan *Scribe: Premkumar K.*

1 Definitions

- $X_1 \cdots X_J = X_{[J]} \in \mathbb{X}_{[J]}$, finite alphabet source.
  $X_{[J]}^n$ iid J–tuple with distribution $p_{X_{[J]}}$.
  $[J]$, first J nodes in N.
- **Sources :** $f_{sj} : \mathbb{X}_s^n \to \mathbb{A}_{sj}^n, s \in [J]$
- **Decoder :** $\displaystyle\bigtimes_{j:jt_0 \in \mathbb{B}} \mathbb{B}_{jt_0}^n \to \left\{ \mathbb{X}_1^n \cdots \mathbb{X}_J^n \right\} \cup \phi$

  Remark 1.

  - *A somewhat unreasonable set up because each node waits for all n inputs before sending information forward.*
  - *An acyclic digraph needed.*

- A network $(N, \mathbb{B}, c)$ is achievable for the source $X_{[J]}$ if $\forall \eta > 0, \lambda \in (0,1)$ there exist $\mathbb{M}_i, i \in N - [J] \cup \{t_0\}$, encoding functions $f_{ij}, i \in N - t_0$, decoding function $g_i, i \in N - [J]$ for the network $(N, \mathbb{B}, c+\eta)$ such that $\Pr\{\widehat{X}_{[J]}^n \neq X_{[J]}^n\} \underset{n}{\leqslant} \lambda$.

Lemma 1. *Let ρ and σ be submodular and super modular rank functions on $[J]$, respectively. There is an $R \in \mathbb{R}_+^J$ such that $\sigma(S) \leqslant R(S) \leqslant \rho(S)$, $\forall S \subseteq [J]$ if and only if $\sigma(S) \leqslant \rho(S), \forall S \subseteq [J]$.*

Proof. $\gamma(S) := \rho(S) - \sigma(S) \forall S \subseteq [J]$ defines a submodular rank function. Let $R \in \mathscr{B}(\gamma)$. Then R is dominated by $\sum_\pi \lambda_\pi R^{(\pi)}$, for some $\{\lambda_\pi\}$ satisfying $\sum_\pi \lambda_\pi = 1, \lambda_\pi \geqslant 0$. $R^{(\pi)}$ are the extreme points of $\mathscr{B}(\gamma)$. Since $\gamma = \rho - \sigma$, we have $R^{(\pi)} = U^{(\pi)} - V^{(\pi)}$ where $U^{(\pi)} \in \mathscr{B}(\rho)$ are the extreme points of $\mathscr{B}(\rho)$ and $V^{(\pi)}$, the extreme points of $\mathscr{C}(\sigma)$.
Let $T = \sum_\pi \lambda_\pi R^{(\pi)} = \sum_\pi \lambda_\pi U^{(\pi)} - \sum_\pi \lambda_\pi V^{(\pi)} - R \geqslant (0, 0, \cdots, 0)$. Then $R = \left(\sum_\pi \lambda_\pi U^{(\pi)} \right) - \left(\sum_\pi \lambda_\pi V^{(\pi)} + T \right) = U - V$ where $U \in \mathscr{B}(\rho)$ and $V \in \mathscr{C}(\sigma)$. Observe that $\mathbf{0} \in \mathscr{B}(\gamma)$, since $\rho \geqslant \sigma$. So $\mathscr{B}(\rho) \cap \mathscr{C}(\sigma) \neq \phi$ $\square$

Remark 2.

Theorem 2. *A network of channels $(N, \mathbb{B}, c)$ is achievable for a source $X_{[J]}$ is and only if $\sigma(S) \leqslant \rho(S), \forall S \subseteq [J]$, where ρ is the min cut capacity function.*

Proof. **Necessity (only if):** Suppose achievability holds with probability $\Pr\left\{ \widehat{X}_{[J]}^n \neq X_{[J]}^n \right\} = \lambda$. The encoding functions $f_{ij}, f_{sj}, g_i, g_{t_0}$ and $X_{[J]}^n$ induce random variables on the input and output of branches $ij \in \mathbb{B}$.
Defining $\lambda(x_{S^c}) := pr\left\{ \widehat{X}_{[J]}^n \neq X_{[J]}^n \middle| X_{S^c}^n = x_{S^c}^n \right\}$, Fano's inequality says $H\left(X_{[J]}^n \middle| \widehat{X}_{[J]}^n, x_{S^c}^n \right) \leqslant 1 + n\lambda(x_{S^c}) \cdot \sum_{k \in S} \log \left| \mathbb{X}_k \right| \leqslant 1 + n\lambda(x_{S^c}) \sum_{j \in [J]} \log \left| \mathbb{X}_j \right| =: r(n, S, x_{S^c}^n)$

Let M_0 be the min cut for S, i.e., $S \subseteq M_0 \subseteq N$ such that $\rho(S) = \min\left\{c(M, N-M)\big|S \subseteq M \subseteq N, t_0 \in N-M\right\} = c(M_0, N-M_0)$. Let $i_1j_1, \cdots, i_qj_q$ be edges along the cut. Let $Y^n(M_0)$ be the inputs to the branches in cut and $Z^n(M_0)$ be the outputs. Given $x^n_{S^c}$, we have the Markov chain, $X^n_{[J]} \longrightarrow Y^n(M_0) \longrightarrow Z^n(M_0) \longrightarrow \widehat{X}^n_{[J]}$
Thus,

$$I\left(X^n_{[J]}; \widehat{X}^n_{[J]}\big|x^n_{S^c}\right) \leqslant I\left(Y^n(M_0); Z^n(M_0)\big|x^n_{S^c}\right) \leqslant n\sum_{k=1}^{q} c_{i_k i_k} = n\rho(S)$$

Putting these together, we get

$$\begin{aligned} H\left(X^n_{[J]}\big|x^n_{S^c}\right) &= I\left(X^n_{[J]}; \widehat{X}^n_{[J]}\big|x^n_{S^c}\right) + H\left(X^n_{[J]}\big|\widehat{X}^n_{[J]}, x^n_{S^c}\right) \\ &\leqslant n\rho(S) + r(n, S, x^n_{S^c}) \end{aligned}$$

Averaging using time independence, we get

$$\sigma(S) \leqslant \rho(S) + \frac{1}{n}\left(1 + n\lambda \sum_{j=1}^{J} \log\left|\mathcal{X}_j\right|\right)$$

Since λ is arbitrary, $\sigma(S) \leqslant \rho(S)$.

□

Proof. **Sufficiency (if) :** By a previous lemma, we can find an $R : \sigma(S) \leqslant R(S) \leqslant \rho(S)$, for every $S \subseteq [J]$. Here is the rest of the argument in an informal fashion.

- First convert to bits. Since $R(S) \geqslant \sigma(S)$, we can find a distributed souce code sequence of rates $\overline{R}_j \leqslant R_j + \delta, j \in [J]$. (Cover's extension to multiple sources). Probability of error $\leqslant \lambda/2$.
- By Ford–Fulkerson, $\exists\ P_{sk}, R_{sk}, s \in [J]; k \in [m_s]$, with $R_s = \sum_{k=1}^{m_s} R_{sk}, s \in [J]$ and $\sum_{P_{sk}:ij \in P_{sk}} R_{sk} \leqslant c_{ij}$. Let $\overline{R}_{sk} = R_{sk} + \frac{\delta}{m_s}$. By choosing δ small enough, we have $\sum_{P_{sk}:ij \in P_{sk}} \overline{R}_{sk} \leqslant c_{ij} + \eta$.
- c_{ij} is achievable on each link.
- Achievability is under an average probability of error criterion. But can show on a single link that capacity does not change for maximal probability of error.
- Chop the SWC bits and route them through paths P_{sk}, for $s \in [J]$. For sufficiently large n, maximal probability of error (across all links) $\leqslant \lambda/2$.
- Overall probability of error $\leqslant \lambda/2 + \lambda/2 = \lambda$.

□