Lecture 21 : Source Coding Networks - Networks with Helpers

Instructor: Rajesh Sundaresan *Scribe: Premkumar K.*

1 Towards a General Theory of Source Networks

Definition:

- A DMMS as before
- $[J]$, $[K]$, $[L]$ sources, intermediate nodes, destinations (graphs of depth 2)
- No edge ends in $[J] = \{j_1, j_2, \cdots, j_J\}$
- No edge starts in $[L] = \{\ell_1, \ell_2, \cdots, \ell_L\}$
- No edges between vertices in $[K] = \{k_1, k_2, \cdots, k_K\}$
- $T_\ell, \ell \in [L] := \{j \in [J] : \text{there is a path from } j \text{ to } \ell\}$
- $D_\ell, \ell \in [L] := \{j \in [J] \cap T_\ell : X_j \text{ must be reproduced at } \ell\}$
- n–length block code of a source network is a family of mappings

$$f_k : \mathbb{X}^n_{\partial_k^-} \to \mathbb{M}_k,\ k \in [K], \quad \text{(encoders)}$$
$$g_\ell : \times_{k \in \partial_\ell^- \cup [K]} \mathbb{M}_k \times \times_{j \in \partial_\ell^- \cup [J]} \to \mathbb{X}^n_{D_\ell},\ \ell \in [L], \quad \text{(decoders)}$$
$$P_e^{(n)}(\ell) := Pr\{\widehat{X}^n_{D_\ell} \neq X^n_{D_\ell}\}$$

- $\frac{\log |\mathbb{M}_k|}{n}$, rate needed at node $k \in [K]$
- $\mathbf{R} = (R_k : k \in [K]) \in \mathbb{R}_+^{[K]}$ is achievable if $\forall \eta > 0, \lambda \in (0,1)$, there exist n–length block codes satisfying

$$\begin{aligned} P_e^{(n)}(\ell) &\underset{n}{\leqslant} \lambda,\ \forall \ell \in [L] \\ \frac{\log |\mathbb{M}_k|}{n} &\leqslant R_k + \eta,\ \forall k \in [K]. \end{aligned}$$

- The set of all these vectors is the achievable rate region.

Remark 1. *The achievable rate region is closed and convex.*

Definitions:

- Normal Source Network: (NSN)
  (1) No edges from inputs $[J]$ to outputs $[L]$.
  (2) $J = K$, and the edges from $[J]$ to $[K]$ define a one-to-one correspondence
  (3) $\partial_\ell^-, \ell \in [L]$ are distinct
  (4) $\partial_{\ell'}^- \subseteq \partial_{\ell''}^- \implies D_{\ell'} \subseteq D_{\ell''}$.

Remark 2.

– *We may refer to a source $X_k, k \in [K]$*

– *Depth = 2*

– ∂_ℓ^- *and* $\partial_{\ell'}^-$ *may have an intersection*

– *If* $\partial_{\ell'}^- \subseteq \partial_{\ell''}^-$, *the demand on* ℓ'' *is more.*

- Helper: $k \in [K]$ is a helper if $\exists\ \ell \in [L]$ such that $k \in \partial_\ell^-$, $\partial_k^- \nsubseteq D_\ell$.
  •$j \in [J]$ is a helper if $\exists \ell \in [L]$ such that $(j\ell)$ is an edge and $j \notin D_\ell$.

Example:
1)

- An NSN
- $D_1 = \{j_1\}$, $D_2 = \{j_3\}$
- $j = 2$ is a helper $\partial_{\ell_1}^- = \{1, 2\}, \partial_{k_2}^- = \{j_2\} \nsubseteq D_{\ell_1} = \{j_1\}$

2) If there are no helpers, $k \in [K]$, $k \in \partial_\ell^- \implies \partial_k^- \subset D_\ell$
3) In an NSN, source nodes are not helpers.

Theorem 1. *(***Csiszar–Korner***) The achievable rate region of an NSN without helpers is the set of* $R \in \mathbb{R}_+^K$ *that satisfy*

$$R(S) \geqslant H(X_S | X_{\partial_\ell^- \} S})$$

for every $\ell \in [L]$, $S \subseteq \partial_\ell^-$, S *is disjoint with every* $\partial_{\ell'}^- \subseteq \partial_\ell$, $\ell \neq \ell'$.

Proof. Converse is clear.
$\forall \ell \in [L]$, we must have $R(S) \geqslant H(X_S | X_{\partial_\ell^- \setminus S})$ $\forall S \subseteq \partial_\ell^-$. Use the Slepian–Wolf–Cover theorem. The statement in theorem ($\forall S \subseteq \partial_\ell^-, S$ disjoint with any $\partial_{\ell'}^- \subseteq \partial_\ell^-$ is a seemingly weaker, but in reality equivalent statement.
•Achievability – Read the paper. They provide a universal code via minimum entropy decoding. □

2 Reduction to NSNs.

Lemma 2. *To every source network N, there exists a source network $\widetilde{N}$ (1) without edges from input to output, (2) having the same set of helpers, (3) achievable rate region of N is a projection of that of $\widetilde{N}$.*

- $\forall j \in [J]$ which is connected to an $\ell \in [L]$
  – introduce $\widetilde{k}, \widetilde{\ell}$ and a path $j\widetilde{k}\widetilde{\ell}$
  – connect $\widetilde{k}$ to all $\ell \in [L]$ which were connected to j
  – Delete all edges originating at j and ending in any destination node.
  – $D_{\widetilde{\ell}} = \{j\}$
- $\widetilde{k}$ cannot be a helper if j were not in N. $\widetilde{k}$ is a helper if j were.
- $\widetilde{R}_k = R_j$, hence a projection.

Lemma 3. *To every source network N without edges from $[J]$ to $[L]$, there is another network $\widetilde{N}$ of the same kind such that*
(1) Edges from input to intermediate nodes of $\widetilde{N}$ are in 1–1 correspondence
(2) The set $[J]$ is a subset of $[\widetilde{J}]$ and R belongs to the achievable rate region of N if and only if

$$\widetilde{R}_k = \begin{cases} R_k & \text{if } k \in [K] \\ 0 & \text{if } k \in [\tilde{K}] \setminus [K] \end{cases}$$

belongs to the achievable rate region of $\widetilde{N}$.
(3) N and $\widetilde{N}$ have the same set of helpers.

Procedure: If X_1 and X_2 are encoded and *decoded* together (always), we may consider X_1X_2 as a new source.
For every $k \in [K]$, consider non empty subsets of $[J]$ of the form $\partial_k^- \cap D_\ell$ for some edge $(k, \ell) \in E$.

$$\left\{\partial_k^-, \partial_k^- \cap D_\ell, \partial_k^- \cap D_{\ell'}, \cdots, \partial_k^- \cap D_{\ell''}\right\}$$

where $(k, \ell), (k, \ell'), \cdots, (k, \ell'')$ are all edges from k to destinations.
Do this for every k. These are all the intermediate nodes and source nodes (1–1 correspondence).

- $[\widetilde{L}] = [L]$ output vertices
- ∂_k^- or $\partial_k^- \cap D_\ell$ is connected with ℓ iff (k, ℓ) is an edge in N
- $Y_{\widetilde{k}} = X_{\partial_k^-}$ or $X_{\partial_k^- \cap D_\ell}$
- D_ℓ should reproduce $Y_{\widetilde{k}}$ if (k, ℓ) is an edge in N and corresponding ∂_k^- or $\partial_k^- \cap D_\ell \subseteq D_\ell$.

For $k_2 : j_2, \{j_2\} \cap \{j_2\}, j_2 \cap j_2, \implies \{j_2\}$. For $k_3 : j_2, j_2 \cap j_2, \implies \{j_2\}$.

Lemma 4. *Given any source network as in the output of Lemma 2, we can change sets $D_\ell, \ell \in [L]$ without changing the achievable rate region and without increasing the number of helpers so that $\partial_\ell^- \subseteq \partial_{\ell'}^- \implies D_\ell \subseteq D_{\ell'}$ for the new network.*
In particular, if there are several outputs with the same ∂_ℓ^-, we can delete all but one, without changimg the achievable rate region.

Procedure: If $\partial_\ell^- \subseteq \partial_{\ell'}^-$, but $D_\ell \nsubseteq D_{\ell'}$, we modify $D_{\ell'}$ to be the union $D_\ell \cup D_{\ell'} \to D_{\ell'}$. A decoder for ℓ will suffice to decode those in the original $D_\ell \diagdown D_{\ell'}$.