Point Process Channel and Capacity of The Exponential Server Queue

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Abstract — **We give a conceptually simple proof for the capacity of the exponential server queue. Our proof links the timing channel to the point-process channel with complete feedback. This point-process approach enables us to bound capacities of timing channels that arise in multiserver queues, queues in tandem, and other simple configurations.**

The capacity of the exponential server queue with service rate μ packets per second is $e^{-1}\mu$ nats per second [1]. The capacity of the point-process channel with maximum input intensity μ points per second, and no background intensity, is also $e^{-1}\mu$ nats per second (cf.[2],[3]). Furthermore, in both channels, the capacity does not increase in the presence of complete feedback. In [1], the connection between both channels in the presence of complete feedback was discussed briefly. In [4], this connection was further explored. It was shown that any strategy on the exponential server channel can be mapped to an equivalent strategy that uses feedback on the point-process channel. This observation implies that the capacity of the exponential server channel is upperbounded by the capacity of the point-process channel with complete feedback, i.e., $e^{-1}\mu$ nats per second.

From [1], we know that $e^{-1}\mu$ nats per second is indeed achievable on the exponential server queue. In other words, although the exponential server queue is only a particular case of a point-process channel with feedback, it attains the point-process channel capacity. In this paper, we provide insight on why there is no loss in capacity.

To see the connection between the queue and the point-process channel, fix a sequence of arrivals denoted by the counting process $x = (x_t : t \in [0,T])$. Let $(Y_t : t \in [0,T])$ be the corresponding counting process of departures from the single-server queue of service rate μ packets per second. Then the state process $(Q_t = x_t - Y_t : t \in [0,T])$ indicates the number of packets in the queue as a function of time. Furthermore, the departure process $(Y_t : t \in [0,T])$ is a self-exciting Poisson process with rate $\lambda = (\lambda_t = \mu 1\{Q_{t-} > 0\} : t \in (0,T])$. Indeed, if $Q_{t-} = 0$, no packet can depart at time t ($\in (0,T]$) and the instantaneous rate of the departure process is 0. If $Q_{t-} > 0$, at least one packet is in the system at $t-$. Due to the memoryless property of exponential service times, the residual time for the next departure is exponentially distributed with mean $1/\mu$ seconds, independent of the past, i.e., the instantaneous rate of the departure process is μ at time t.

It is well-known that the sample function density (which plays the role of probability density) given input x, is $p(x,y)$, where

$$p(x,y) \triangleq \exp\left\{\int_0^T [\log(\lambda_t)\ dy_t - \lambda_t\ dt]\right\}. \qquad (1)$$

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Furthermore, for a given probability measure on the input space, the normalized mutual information is

$$\frac{1}{T} I_T(X;Y) = \frac{1}{T} E \int_0^T dt\ \left[\phi(\lambda_t) - \phi\left(\hat{\lambda}_t\right)\right], \qquad (2)$$

where $\hat{\lambda}_t = E[\lambda_t | (Y_s : s \in [0,t))]$, for each $t \in [0,T]$, and $\phi(u) = u \log u$, (see [2], [3], [5]). We take $\phi(0) = 0$. Note that $\hat{\lambda}_t$ is an estimate of the rate of the departure process given prior departures.

We can show the existence of codes that have vanishing probability of error (as the observation interval T increases without bound) at rate $e^{-1}\mu$ nats per second. Here, for brevity, we only argue that there is an input probability measure such that the normalized mutual information equals the upperbound $e^{-1}\mu$ nats per second. The input measure should induce the following properties to attain the upperbound.

(*a*) $\lambda_t = 0$ or μ.
(*b*) $(1/T)\int_0^T dt\ E[\lambda_t] = e^{-1}\mu$.
(*c*) λ_t should be independent of prior departures $(Y_s : s \in [0,t))$, and $E[\lambda_t]$ should be a constant over time, i.e., $\hat{\lambda}_t = e^{-1}\mu$.

Let the input probability measure be a Poisson process with rate $e^{-1}\mu$ packets per second. Let the queue be in equilibrium at $t = 0$. We then have an $M/M/1$ queueing system. Property (*a*) holds because λ_t is μ times an indicator function. Property (*b*) follows from ergodicity of the state process and the fact that the queue is nonempty with probability e^{-1}. Property (*c*) holds by Burke's theorem (for e.g., [5, V.T1]); the state of the queue Q_t is independent of prior departures $(Y_s : s \in [0,t))$ and therefore so is λ_t.

The point-process approach via (1), (2) and the filtering techniques of [5] (to provide estimates of queue size) can be used to find achievable rates of some simple networks of exponential servers. In [6], lower bounds on the capacities of multiserver queues and two queues connected in tandem are provided.

References

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