On The Duality Between Rate And Power Optimizations

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***Abstract*— Sequence design problems are considered in this paper. The problem of sum power minimization in a spread spectrum system can be reduced to the problem of sum capacity maximization, and vice versa. A solution to one of the problems yields a solution to the other. Subsequently, conceptually simple sequence design algorithms known to hold for the white-noise case are extended to the colored noise case. The algorithms yield an upper bound of $2N - L$ on the number of sequences where N is the processing gain and L the number of non-interfering subsets of users. If some users (at most $N - 1$) are allowed to signal along a limited number of multiple dimensions, then N orthogonal sequences suffice.**

I. Introduction

Consider a symbol-synchronous code-division multiple access (CDMA) system. The kth user is assigned an N-sequence $s_k \in \mathbb{R}^N$ of unit energy, i.e., $s_k^t s_k = 1$. The processing gain is N chips, and the number of users is K. User k modulates the vector s_k by its data symbol $X_k \in \mathbb{R}$ and transmits $X_k s_k$ over N chips. This transmission interferes with other users' transmissions and is corrupted by noise. The received signal is modeled by

$$Y = \sum_{k=1}^{K} s_k X_k + Z,$$

where Z is a zero-mean Gaussian random vector with a covariance matrix Σ. We will consider the following two sequence design problems.

Problem I: User k has a power constraint p_k units per chip, i.e., $E[X_k^2] \leq N p_k$. The goal then is to assign sequences and data rates to users so that the sum of the individual rates at which the users can transmit data reliably (in an asymptotic sense) is maximized. The maximum value C_{sum} is called the sum capacity.

Problem II, a dual to Problem I, is one where user k demands reliable transmission at a minimum rate r_k bits/chip. The goal is to assign sequences and powers to users so that despite their mutual interference and noise, each of the users can transmit reliably at or greater than their required rates, and the sum of the received powers (energy/chip) at the base-station is minimized.

Viswanath and Anantharam [1] have solved Problem I and provided an explicit characterization for the sum capacity. Guess [2] has solved Problem II for the particular case when $\Sigma = I_N$. The similarity of characterizations of the solutions to Problems I and II when $\Sigma = I_N$ suggests a close relationship between the two problems. In this paper, we show that the two problems can be reduced to a single optimization problem by establishing a mapping between them. Furthermore, we extend the algorithm of [3] to cover the colored noise case. In particular, we show that $2N - L$ sequences are sufficient, where L is the number of non-interfering subsets of users. By allowing a mild splitting, we show that N orthogonal sequences are optimal. This greatly simplifies signaling of parameters on the downlink to achieve optimality on the uplink.

Since the work of Rupf and Massey [4], rate maximization and power minimization for multiple access systems have attracted considerable attention - Viswanath and Anantharam [5], Guess [2], among other papers in the CDMA setting, and Wunder and Michel [6] in the OFDM setting. Given a set of rates whose sum is r_{tot}, Guess [2] shows the minimum sum power required to achieve this set of rates is same as the sum of power constraints that yield a sum capacity equal to r_{tot}. In [2], Guess also provides expressions for the set of all achievable rate-tuples with a constraint on the sum power and the set of all admissible power assignments to achieve a target sum capacity. Our duality result goes a step further and shows that a solution to one of the problems can be mapped to a solution on the other. Tse and Hanly [7] indicate that the rate maximization and power minimization for the no spreading case are duals of each other by making use of the polymatroid - contra-polymatroid duality. We show that a similar duality holds with spreading. This suggests a similar unity between such problems in all settings, CDMA or otherwise.

Several algorithms to identify an optimal set of sequences have been proposed. Viswanath and Anantharam [5] provide a recursive algorithm to identify sequences which achieve sum capacity. The configuration that attains sum capacity also minimizes a potential-like quantity called total squared correlation (TSC). Anigstein and Anantharam [8] provide an iterative algorithm suitable for distributed implementation. The algorithm converges to the configuration that minimizes the TSC. Our focus here is on finite-step algorithms. We refer the reader to the work of Tropp and others [9] and references therein for a summary of several finite-step algorithms. We provide another finite-step algorithm that has an interesting water-filling interpretation.

This paper is organized as follows. In Section II we reduce Problem II to Problem I and show how a solution to Problem I can be mapped to a solution to Problem II. Section III extends the sequence design algorithms for colored noise. In Section IV, we prove that $2N-L$ sequences for single dimensional signaling and N sequences for multidimensional signaling are sufficient to achieve optimality on the colored noise channel.

II. The Duality

In this section, we reduce Problem II to Problem I. We begin with some preliminaries.

A. Preliminaries

Suppose user k is assigned sequence s_k and is received at power p_k energy/chip. Let S be the $N \times K$ matrix $[s_1 \ s_2 \ \cdots \ s_K]$ and $P = \text{diag}(p_1, p_2, \cdots, p_K)$. For any $J \subseteq \{1, \cdots, K\}$, we denote the $N \times |J|$ matrix $[s_j; \ j \in J]$ by S_J and the $|J| \times |J|$ matrix $\text{diag}(p_j; \ j \in J)$ by P_J. Then, the capacity region can be written as (see, for example, [1])

$$C(S,P,\Sigma) = \bigcap_{J \subset \{1,\cdots,K\}} \Big\{ (r_1, \cdots, r_K) \in \mathbb{R}_+^K : \sum_{k \in J} r_k \leq \frac{1}{2N} \log \left| I_N + N\Sigma^{-1} S_J P_J S_J^t \right| \Big\},$$

where $|A|$ denotes the determinant of the matrix A and r_k is user k's data rate in nats/chip. In this paper all logarithms are natural logarithms.

Definition 1: Let $\mathcal{S}$ denote the set of all sequence matrices

$$\mathcal{S} = \{ S \in \mathbb{R}^{N \times K} : \left(S^t S\right)_{ii} = 1 \text{ for } i = 1, 2, \cdots, K \},$$

and $\mathcal{P}$ the set of all power allocation policies

$$\mathcal{P} = \{ P \in \mathbb{R}_+^{K \times K}, P \text{ diagonal} \}.$$

(a) We say that the pair (S,P) is *valid* if $S \in \mathcal{S}$ and $P \in \mathcal{P}$.
(b) Given a rate vector $r \in \mathbb{R}_+^K$, we say (S,P) is a *design for* r *on* Σ if (S,P) is valid and $r \in C(S,P,\Sigma)$.
(c) We say (S,P) is a *commuting design for* r *on* Σ if (S,P) is a design for r on Σ and SPS^t and Σ commute.

□

We can now formulate Problem II as follows.

Problem II : Given $r \in \mathbb{R}_+^K$ and a positive definite matrix Σ find

(a) $P_{\min} = \min\{\text{trace}(P) : (S,P) \text{ is a design for } r \text{ on } \Sigma\}$, and

(b) the (S,P) that attains the minimum. □

Guess [2] proves that the minimum exists for $\Sigma = I_N$. We will show existence for any positive definite Σ.

Let $\text{eig}\,(A)$ denote the eigenvalues of a Hermitian matrix A in the decreasing order of their values and $\text{eig}_i\,(A)$ its ith component. Observe that

$$\begin{aligned} \text{trace}\,(P) &= \text{trace}\left(PS^tS\right) = \text{trace}\left(SPS^t\right) \\ &= \text{trace}\left(\Sigma + SPS^t\right) - \text{trace}\,(\Sigma) \\ &= \sum_{i=1}^{N} \text{eig}_i\left(\Sigma + SPS^t\right) - \text{trace}\,(\Sigma). \end{aligned}$$

Hence, Problem II can be restated as follows : Identify

(a) the vector in the set

$$\{\text{eig}(\Sigma + SPS^t) : (S,P) \text{ is a design for } r \text{ on } \Sigma\}$$

with the minimum sum; and

(b) the pair (S,P), a design for r on Σ, that achieves the minimum.

As observed in many sequence design problems, majorization turns out to be a useful tool to characterize this set.

B. Majorization

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $x_{[1]}, \cdots, x_{[n]}$ denote the components of x in decreasing order, i.e., $x_{[1]} \geq \cdots \geq x_{[n]}$.

Definition 2: Let $x, y \in \mathbb{R}_+^n$. We say x *product majorizes* y and denote the relation by $x \overset{\otimes}{\succ} y$ if

$$\textstyle\prod_{i=1}^{j} x_{[i]} \quad \geq \quad \prod_{i=1}^{j} y_{[i]}$$

for $j = 1, \cdots, n$, with equality when $j = n$.

For $x, y \in \mathbb{R}^n$, we say x *sum majorizes* y and denote by $x \overset{\oplus}{\succ} y$ if

$$\textstyle\sum_{i=1}^{j} x_{[i]} \quad \geq \quad \sum_{i=1}^{j} y_{[i]}$$

for $j = 1, \cdots, n$, with equality when $j = n$. □

For $a \in \mathbb{R}_+^n$ and $b \in \mathbb{R}^n$, define the vector

$$\log a \triangleq (\log a_1, \cdots, \log a_n)$$

and the vector

$$e^b \triangleq (e^{b_1}, \cdots, e^{b_n}).$$

Note that if $x, y \in \mathbb{R}_+^n$, then

$$x \overset{\otimes}{\succ} y \Rightarrow \log x \overset{\oplus}{\succ} \log y.$$

Conversely, if $x, y \in \mathbb{R}^n$, then

$$e^x \overset{\otimes}{\succ} e^y \Rightarrow x \overset{\oplus}{\succ} y.$$

Definition 3: A function $f : \mathbb{R}^n \to \mathbb{R}$ is *product-Schur convex* if

$$x \overset{\otimes}{\succ} y \Rightarrow f(x) \geq f(y).$$

A function $g : \mathbb{R}^n \Rightarrow \mathbb{R}$ is *sum-Schur convex* if

$$x \overset{\oplus}{\succ} y \Rightarrow g(x) \geq g(y).$$

A function h is Schur concave if $-h$ is Schur convex in the appropriate sense. □

C. The Reduction

Our goal in this subsection is to reduce Problem II to that of Problem I. We do this by first identifying the following relevant set that yields commuting designs. Let $\text{eig}(\Sigma) = (\sigma_N^2, \cdots, \sigma_1^2)$. Define

$$\mathbb{L} = \left\{ \mu \in \mathbb{R}^N : \begin{array}{c} \frac{\mu_i}{\sigma_i^2} \geq 1,\ i = 1, \cdots, N, \\ \prod_{i=1}^{j} \frac{\mu_i}{\sigma_i^2} \geq \prod_{i=1}^{j} e^{2Nr_{[i]}}, 1 \leq j < N, \\ \prod_{i=1}^{N} \frac{\mu_i}{\sigma_i^2} = \prod_{i=1}^{K} e^{2Nr_{[i]}}. \end{array} \right\}$$

Lemma 4: For every $\mu \in \mathbb{L}$, there exists a commuting design (S, P) for r on Σ with $\mu = \text{eig}(SPS^t + \Sigma)$. □

We provide only an outline of a proof. We first identify a design for r on I_N. Then we apply a capacity-preserving unitary transformation to the sequences to get another design for r on I_N that in addition commutes with the given Σ. We then apply another capacity preserving transformation to arrive at a commuting design for r on Σ.

We next show that we may restrict our search to commuting designs.

Lemma 5: If (S, P) is a design for r on Σ, there exists a commuting design $(\tilde{S}, \tilde{P})$ for r on Σ that satisfies trace$(\tilde{P}) \leq$ trace(P) □

In order to prove this lemma, we start from the design (S, P) for r on Σ and apply a sequence of capacity-preserving transformations to designs on I_N and then transform the result back to a design on Σ that matches the eigenvectors of the received signal covariance matrix with those of Σ. The eigenvalues of the signal matrix in increasing order match with those of Σ in decreasing order, a property that makes the same capacity region achievable with a reduced power.

The following lemma now highlights the importance of $\mathbb{L}$.

Lemma 6: If (S, P) is a commuting design for r on Σ, then there exists a $\mu \in \mathbb{L}$ such that $\sum_{i=1}^{N} \mu_i \leq$ trace$(\Sigma + SPS^t)$. □

Thus, every vector in $\mathbb{L}$ leads to a commuting design for r on Σ and conversely, every design for r on Σ can be replaced by a possibly better commuting design for r on Σ whose eig$(\Sigma + SPS^t)$ belongs to $\mathbb{L}$. It is therefore sufficient to search for an optimal design within this set. Hence we have the following partial restatement of Problem II.

Theorem 7: $P_{min} = \min \left\{ \sum_{i=1}^{N} \mu_i : \mu \in \mathbb{L} \right\} -$ trace(Σ). □

This theorem easily follows from Lemmas 4, 5, and 6.

The set $\mathbb{L}$ is related to a set $\mathbb{L}'$ whose sum-Schur minimal element yields the sum capacity value in the power constrained problem studied by Viswanath and Anantharam in [1]. $\mathbb{L}'$ is given by

$$\mathbb{L}' = \left\{ l \in \mathbb{R}^N : \begin{array}{c} l_i - \eta_i^2 \geq 0, i = 1, \cdots, N, \\ \sum_{i=1}^{j} l_i - \eta_i^2 \geq N \sum_{i=1}^{j} p_{[i]}, \\ 1 \leq j < N, \\ \sum_{i=1}^{N} l_i - \eta_i^2 = N \sum_{i=1}^{K} p_{[i]}. \end{array} \right\} \quad (1)$$

We now show the precise duality between the power minimization and sum capacity maximization problems. Without loss of generality, we may assume $\sigma_i^2 > 1$ for $1 \leq i \leq N$.

Theorem 8: Given the vectors r and $\sigma^2 = (\sigma_1^2, \cdots, \sigma_N^2)$, if $\mu \in \mathbb{L}$, then $l = \log \mu \in \mathbb{L}'$ with $\eta^2 = \log \sigma^2$ and $p = 2r$. Furthermore, the sum power minimization can be written as

$$P_{min} = \min \left\{ \sum_{i=1}^{N} e^{l_i} : l \in \mathbb{L}' \right\} - \text{trace}(\Sigma).$$

Conversely, for a given η^2 and p, if $l \in \mathbb{L}'$, then $\mu = e^l \in \mathbb{L}$ with $\sigma^2 = e^{\eta^2}$ and $r = p/2$. □

It is well-known that $g(l) = \sum_{i=1}^{N} e^{l_i}$ is a sum-Schur convex function. If we can identify a sum-Schur minimal element l^* of $\mathbb{L}'$, then $P_{min} = g(l^*) -$trace(Σ). Moreover the corresponding element e^{l^*} would be a product-Schur minimal element of $\mathbb{L}$. The existence of the sum-Schur minimal element of $\mathbb{L}'$ is established by means of a finite-step algorithm in [1]. We thus have the following reduction.

Theorem 9: The minimum value P_{min} is given by $g(l^*) -$ trace(Σ), where l^* is the sum-Schur minimal element of the set $\mathbb{L}'$ given in (1). □

A similar reduction of the sum capacity maximization problem to the sum power minimization problem holds.

Having found the minimum sum power value, we now show how to find a valid design for r on Σ that achieves this minimum value.

III. Sequence Design Algorithms

In this section, we describe an algorithm to get a commuting design (S, P) for r on Σ. The proposed algorithm is a finite-step algorithm which is numerically stable. There are several other finite-step algorithms [1], [9], but ours admits a water-filling interpretation. Moreover, it provides an upper bound on the number of required sequences in a rather direct fashion.

Definition 10: We say Σ *and* r *admit a water-filling solution* if the product-Schur minimal element μ^* of $\mathbb{L}$ has equal components. □

Without loss of generality, we may assume that Σ is diagonal. If Σ and r do not admit a water-filling solution, from the results of Viswanath and Anantharam [1], the optimal solution partitions the set of users and associates a disjoint set of dimensions for each subset of the partition. The r and Σ restricted to any one subset of the partition does indeed admit a water-filling solution over its associated set of dimensions. We may thus assume for purposes of sequence allocation that r and Σ admit a water-filling solution.

We now describe the intuition behind Algorithm 11 below. The algorithm is a generalization of the algorithm for the white noise case given in [3]. One of the facets of the water-filling solution in Definition 14 is that every user can be accommodated in the least noisy dimension in the absence of other users. We assign sequences to users in a sequential fashion. At any stage, the algorithm attempts to fit the user in the noisiest dimension. If this user's rate requirement forces us to exceed the water filling level in this dimension, the algorithm pours the remaining energy in the next noisy dimension, and so on. Note that this pouring has to be done within the constraint of the interlacing inequality, i.e., the new set of eigenvalues $\lambda^{(k)}$ after the user k is added satisfies

$$\lambda_1^{(k)} \geq \lambda_1^{(k-1)} \geq \lambda_2^{(k)} \geq \lambda_2^{(k-1)} \geq \cdots \geq \lambda_N^{(k)} \geq \lambda_N^{(k-1)}.$$

The results of [3, Prop. 1] guarantee the existence of a subroutine c that takes as input a matrix A with eigenvalues $\lambda^{(k-1)}$ and a set of prescribed eigenvalues $\lambda^{(k)}$ that interlace $\lambda^{(k-1)}$, and outputs $c = c\left(A, \lambda^{(k)}\right)$ such that $\text{eig}(A + cc^*) = \lambda^{(k)}$. We use this subroutine at each execution of Step 3 in the following algorithm to put out a sequence.

Algorithm 11: Algorithm for Problem II

- **Inputs**: $K \leftarrow$ number of users, $N \leftarrow$ processing gain, $r \leftarrow$ user rate requirements, $(\sigma_N^2, \cdots, \sigma_1^2) \leftarrow \text{eig}(\Sigma)$.
- **Initialization**: Set

$$\lambda_{\max} \leftarrow \exp\left\{2\sum_{k=1}^{K} r_k\right\} \left(\prod_{i=1}^{N} \sigma_i^2\right)^{\frac{1}{N}},$$

  Set $\lambda_n^{(k)} \leftarrow \sigma_n^2$ for $k = 0, 1, \cdots, K$, and $n = 1, \cdots, N$ and $\lambda_{N+1}^{(k)} \leftarrow \lambda_{\max}$ for $k = 0, 1, \cdots, K$. Set the user index $k \leftarrow 1$, $d \leftarrow N$. (The quantity d indexes the dimension with the highest noise plus interference at the current stage of design with value less than $\lambda_{\max}$). Set $A_0 \leftarrow V$, the unitary matrix that diagonalizes Σ.
- **Step 1**: If $k > K$, stop.
- **Step 2**: Let

$$n^*(k) = \arg\max_{1 \leq n \leq N} \left\{\lambda_n^{(k-1)} \exp\{2Nr_k\} \leq \lambda_{\max}\right\}$$

  indicate the most noisy dimension in which user k fits.
- **Case 2(a)**: If $n^*(k) = d$ and $\lambda_d^{(k-1)} \exp\{2Nr_k\} = \lambda_{\max}$, then set $\lambda_d^{(j)} \leftarrow \lambda_{\max}$ for $j = k, k+1, \cdots, K$, and $d \leftarrow d - 1$. Go to **Step 3**.
- **Case 2(b)**: If $n^*(k) = d$ and

$$\lambda_d^{(k-1)} \exp\{2Nr_k\} < \lambda_{\max},$$

  then set

$$\lambda_d^{(j)} \leftarrow \lambda_d^{(k-1)} \cdot \exp\{2Nr_k\} \text{ for } j = k, k+1, \cdots, K.$$

  Go to **Step 3**.
- **Case 2(c)**: Else if $n^*(k) < d$ and

$$\lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2Nr_k\} = \lambda_{\max},$$

  then set $\lambda_n^{(j)} \leftarrow \lambda_{n+1}^{(k-1)}$ for $n = d, d-1, \cdots, n^*(k), j = k, \cdots, K$, and $d \leftarrow d - 1$. Go to **Step 3**.
- **Case 2(d)**: Else if $n^*(k) < d$, and

$$\lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2Nr_k\} < \lambda_{\max},$$

  then set $\lambda_n^{(j)} \leftarrow \lambda_{n+1}^{(k-1)}$ for $n = d, d-1, \cdots, n^*(k)+1$, $j = k, \cdots, K$, and

$$\lambda_{n^*(k)}^{(j)} \leftarrow \lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2Nr_k\} \cdot \lambda_{n^*(k)+1}^{(k-1)} / \lambda_{\max}.$$

  for $j = k, \cdots, K$. Also set $d \leftarrow d - 1$.
- **Step 3**: Identify the vector $c_k = c\left(A_{k-1}, \lambda^{(k)}\right)$. Then set $s_k \leftarrow c_k / ||c_k||$, $p_k \leftarrow (c_k^t c_k)/N$. This provides the sequence and power for user k. Finally, set $A_k \leftarrow A_{k-1} + c_k c_k^t$, $k \leftarrow k + 1$, and go to **Step 1**.

□

A similar generalization holds for the sum capacity maximizing sequence allocation algorithm. The proof of validity of these algorithms is omitted for brevity.

Algorithm 11 has a complexity of $O(KN)$ floating point operations and is numerically stable. This fact is proved for [3, Algorithm 3] and can be extended to Algorithm 11. Tropp and others [9] also provide a numerically stable sequence design algorithm for Problem I whose complexity is $O(KN)$ floating point operations. An interesting property of Algorithm 11 is that it guarantees optimality with at most $2N-1$ sequences, as discussed in the next section, and will work for any ordering of users.

IV. Bounds On Number of Sequences and Multi-Dimensional Signaling

In this section we study the number of sequences for optimal design.

A. $2N - 1$ *sequences suffice*

We first argue, using the water-filling interpretation, that $2N - 1$ sequences suffice regardless of the number of users K. If $K > 2N - 1$, then some users share the same sequence and hence will completely overlap with each other. But a successive cancellation receiver enables us to receive data from all such users if powers or rates are suitably assigned.

Theorem 12: There is an optimal sequence allocation with at most $2N - 1$ distinct sequences for both Problems I and II. Furthermore, if the Schur-minimal element results in L non-interfering subsets of users, then there is an optimal sequence allocation with at most $2N - L$ distinct sequences. □

To see this, observe that in Algorithm 11, at most two eigenvalues differ in value after the addition of a user. A new sequence is put out when either a dimension gets exceeded, or when a first step is made in a new dimension. Since there are at most $N - 1$ possible crossings, and N first steps, the number of sequences is upper bounded by $2N - 1$. If r and Σ do not admit a water-filling solution, then an execution of this algorithm for the non-interfering subset j over its associated N_j dimensions yields at most $2N_j - 1$ distinct sequences. Summing them across L partitions we get the upper bound $2N - L$.

B. Two dimensional signaling and sufficiency of N orthogonal signals

We now show how to achieve sum capacity (respectively, minimum sum power) by using at most N orthogonal sequences. The above algorithms confine each user to signal along a single dimension. It is possible in some cases that Algorithm 11 leads to a set of orthogonal sequences. The following example illustrates this.

Example 13: Consider $N = 3$, $\Sigma = \text{diag}(8, 5, 2)$, and $K = 5$. Let the five users have power constraints $p = (p_1, p_2, p_3, p_4, p_5) = (3, 2, 2, 4, 1)$ units/chip. Allocate rates and sequences to these users so that the sum rate is maximized. □

Assigning sequences in the increasing order of their indices, Algorithm 11 results in the following sequence assignment. User 1 is assigned the sequence $(1, 0, 0)^t$, users 2 and 3 share the sequence $(0, 1, 0)^t$, and users 4, 5 share $(0, 0, 1)^t$. Hence in this example we achieve sum capacity through an orthogonal sequence assignment. The key to attaining orthogonality is the fact that on adding each user exactly one eigenvalue changes. This property is captured precisely in the following definition.

Definition 14: Let $x \in \mathbb{R}_+^K$ and $y \in \mathbb{R}^N$. The vector x *has a water-filling partition of size* N *over* y if there is a partition of the users $\{1, 2, \cdots, K\}$ into N subsets $S_1, S_2, \cdots, S_N$, such that

$$\sum_{k \in S_n} x_k + y_n \; = \; \frac{\sum_{k=1}^K x_k + \sum_{n=1}^N y_n}{N},$$

for $n{=}1, 2, \cdots, N$. The subsets $S_1, S_2, \cdots, S_N$ will be referred to as the *water filling partition*. □

If the rate vector r has a water filling partition of size N over $\frac{1}{2N} \log(\text{eig}(\Sigma))$, then Σ and r admit a water filling solution. Analogously if the power constraint vector p has a water filling partition of size N over $\text{eig}(\Sigma)$, then Σ and p admit a water filling solution.

Proposition 15: If the rate vector r has a water filling partition of size N over $\frac{1}{2N} \log(\text{eig}(\Sigma))$, then N orthogonal sequences are sufficient to attain the minimum sum power, which is given by

$$P_{min} = \exp\left\{2 \sum_{k=1}^K r_k\right\} \left(\prod_{i=1}^N \sigma_i^2\right)^{\frac{1}{N}} - \frac{\sum_{i=1}^N \sigma_i^2}{N}. \qquad (2)$$

Analogously if the power constraint vector p has a water filling partition of size N over $\text{eig}(\Sigma)$, then N orthogonal sequences are sufficient to attain the sum capacity given by

$$C_{sum} = \frac{1}{2} \log\left(p_{tot} + \frac{\sum_{n=1}^N \sigma_n^2}{N}\right) - \frac{1}{2N} \log\left(\prod_{n=1}^N \sigma_n^2\right). \qquad (3)$$

□

An execution of the algorithm that assigns sequences and powers to all users in subset S_N before all users in subset S_{N-1}, and so on, results in an orthogonal allocation. This is because each subset fills exactly one dimension, after taking into account the noise level in that dimension. The proposition can also be proved directly. For user $k \in S_n$, assign the sequence e_n. This will ensure users from across subsets do not cause interference to each other. Power assigned to user k is exactly equal to that needed to meet his rate requirement via a successive interference cancellation scheme. It can be shown that this orthogonal allocation achieves minimum sum power. We omit the details.

Having derived a sufficient condition for optimality of N orthogonal sequences, we now discuss how to obtain this condition from any arbitrary set of rate requirements. An analogous statement holds for a set of power constraints.

Proposition 16: If Σ and r admit a water-filling solution, then r can be cast into a vector r' of size K' virtual users where $K \le K' \le K + N - 1$, and r' has a water-filling partition of size N over $\frac{1}{2N} \log\ \text{eig}(\Sigma)$. Moreover r' is obtained by splitting $K' - K$ users into exactly two virtual users each. □

Users are allocated sequences in decreasing order of their rates. Power is poured to the required level in dimensions that go in increasing order of noise power. It can be shown that such a procedure ensures that a split user uses at most two dimensions and therefore at most two sequences. Moreover, at most $N - 1$ users are split into two virtual users each.

If Σ and r do not admit a water-filling solution, at most $N-1$ splits are required for a water-filling optimal solution. Some users may signal along multiple dimensions. The resulting set of sequences is orthogonal and of size N.

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