An Analog MVUE for a Wireless Sensor Network

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***Abstract*— An analog minimum-variance unbiased estimator (MVUE) over an asymmetric wireless sensor network is studied. Minimisation of variance is cast into a constrained non-convex optimisation problem. An explicit algorithm that solves the problem is provided. The solution is obtained by decomposing the original problem into a finite number of convex optimisation problems with explicit solutions. These solutions are then juxtaposed together by exploiting further structure in the objective function.**

I. Introduction and Problem Statement

In this paper, we study a distributed analog minimum variance unbiased estimator (MVUE) over an asymmetric wireless sensor network. In a typical sensor network, sensors communicate their observations with a central node via a scheduled transmission or via a random access method with a collision resolution mechanism. It is usually also the case that these exact observations themselves are not as important as a function of these observations. Furthermore, when the observations at sensors are corrupted by Gaussian noise, the desired function at the central node is a sum of the observations. Observing that the wireless multiple-access channel enables superposition of simultaneous transmissions, we propose a physical layer fusion mechanism in this paper.

The setting is as follows.

- The network has L sensors that make observations of an underlying parameter $\theta \in \mathbb{R}$. The observations X_l at the lth sensor are noisy and are modeled as random variables with the Gaussian distribution of mean θ and observation noise variance σ^2_{obs}, i.e., $X_l \sim \mathcal{N}(\theta, \sigma^2_{\mathsf{obs}})$.
- The sensors transmit their observations over a Gaussian multiple-access channel (GMAC) in an analog fashion. Specifically, the lth sensor transmits
$$Y_l = \alpha_l X_l = \alpha_l \theta + \alpha_l Z_l,$$
where $Z_l \sim \mathcal{N}(0, \sigma^2_{\mathsf{obs}})$.
- The scaling factor α_l is constrained by $0 \le \alpha_l \le \alpha_{\max}$, as would be the case when there is a power constraint.
- The deterministic channel gain from the lth sensor to the fusion centre is $h_l \in \mathbb{R}_+$ and is assumed to be known.
- The channel output is
$$\tilde{Y} = \sum_{l=1}^{L} h_l Y_l + Z_{\mathsf{MAC}}$$
where $Z_{\mathsf{MAC}} \sim \mathcal{N}(0, \sigma^2_{\mathsf{MAC}})$.
- Given θ, the random variables X_l are independent of each other. Furthermore, the GMAC noise Z_{MAC} is independent of all other random variables.

Observe that in this distributed setting with analog transmission, we can get an unbiased estimate $\hat{\theta}$ from $\tilde{Y}$ using the bijective transformation
$$\hat{\theta} := \frac{\tilde{Y}}{\sum_{l=1}^{L} h_l \alpha_l} \sim \mathcal{N}(\theta, \sigma^2)$$
where
$$\sigma^2 := \frac{\sigma^2_{\mathsf{obs}} \sum_{l=1}^{L} h_l^2 \alpha_l^2 + \sigma^2_{\mathsf{MAC}}}{\left(\sum_{l=1}^{L} h_l \alpha_l\right)^2}$$
is the resulting noise variance of the analog estimator. This variance is a combination of both the observation and GMAC channel noise variances. Our goal is to find the analog scaling factors $\alpha_l, l = 1, \cdots, L$ that minimise the variance of this unbiased estimator. More precisely,

Problem 1:
$$\text{Minimise} \quad \frac{\sigma^2_{\mathsf{obs}} \sum_{l=1}^{L} h_l^2 \alpha_l^2 + \sigma^2_{\mathsf{MAC}}}{\left(\sum_{l=1}^{L} h_l \alpha_l\right)^2}$$
$$\text{subject to} \quad 0 \le \alpha_l \le \alpha_{\mathsf{max}}, \quad l = 1, \ldots, L.$$
□

Remarks:

- We consider only a single-shot estimation. If θ_k varies over time, single-shot estimation is optimal when $\{\theta_k\}$ is an independent process.
- The observation noise variance σ^2_{obs} is taken to be the noise variance in one observation sample, even though it may be the result of a local smoothing via multiple samples at the same sensor at a higher rate of sensing.
- The same optimisation also arises in a stochastic optimal control problem for the detection of a change in a sensor network with minimum detection delay. See [1] for details.
- The symmetric case when $h_l = 1, l = 1, \cdots, L$ is easy to solve. By Cauchy-Schwarz inequality,
$$\left(\sum_l h_l \alpha_l\right)^2 \le L \sum_l h_l^2 \alpha_l^2$$

so that
$$\frac{\sigma^2_{\mathsf{obs}} \sum_l h_l^2 \alpha_l^2}{\left(\sum_l h_l \alpha_l\right)^2} \geq \frac{\sigma^2_{\mathsf{obs}}}{L}$$
with equality when $h_l \alpha_l$ does not vary with l. The second term in the variance $\frac{\sigma^2_{\mathsf{MAC}}}{\left(\sum_l h_l \alpha_l\right)^2}$ is minimized when $\alpha_l = \alpha_{\max}$ for every l. Both these requirements are met when $h_l = 1$ and therefore $\alpha_l = \alpha_{\max}$ for $l = 1, \cdots, L$ solves the problem in the symmetric case. This special case was considered in [2] in a sequential change detection setting.

- In the asymmetric case, the two requirements cannot be simultaneously met making the optimisation problem an interesting one.

II. Solution

We first give an explicit algorithm that identifies the optimal α. We then give a proof of its optimality.

Algorithm 1: Let $h_1 \leq \cdots \leq h_L$.

- **Step 1**: Find the least $k \in \{1, \ldots, L-1\}$ that satisfies
$$h_k \sum_{l=1}^{k} h_l \leq \sum_{l=1}^{k} h_l^2 + \frac{\sigma^2_{\mathsf{MAC}}}{\sigma^2_{\mathsf{obs}} \alpha^2_{\mathsf{max}}} \leq h_{k+1} \sum_{l=1}^{k} h_l. \quad (1)$$
If this is not satisfied for any such k, put $k = L$.
- **Step 2**: Set
$$a^* = \alpha_{\mathsf{max}} \sum_{l=1}^{k} h_l + \frac{(L-k)\alpha_{\mathsf{max}} \sum_{l=1}^{k} h_l^2}{\sum_{l=1}^{k} h_l} + \frac{\sigma^2_{\mathsf{MAC}}(L-k)}{\sigma^2_{\mathsf{obs}} \alpha_{\mathsf{max}} \sum_{l=1}^{k} h_l}. \quad (2)$$
- **Step 3**: The optimal α is given by
$$\begin{aligned} \alpha_m &= \alpha_{\mathsf{max}}, \ 1 \leq m \leq k, \\ \alpha_m &= \frac{a^* - \alpha_{\mathsf{max}} \sum_{l=1}^{k} h_l}{(L-k) h_m}, \quad k < m \leq L. \end{aligned} \quad (3)$$

□

Thus the optimal choice sets amplitudes of the k sensors with the k worst channels to α_{max}. The remaining sensors' amplitudes are appropriately chosen smaller values. Intuitively, sensors $l = k+1, \ldots, L$ have so good a channel that scaling by $\alpha_{\max}$ for these sensors will amplify the observation noise leading to a larger overall noise variance. Note that when all channel gains are equal, we recover $\alpha_l = \alpha_{\mathsf{max}}$ for all sensors, as remarked in Section I.

We take $h_1 \leq \cdots \leq h_L$. To solve Problem 1, we add the constraint $\sum_{l=1}^{L} h_l \alpha_l = a$, where without loss of generality $a \in [0, a_{\mathsf{max}}]$, with $a_{\mathsf{max}} = \alpha_{\mathsf{max}} \sum_{l=1}^{L} h_l$, and solve the convex optimization problem:

Problem 2:
$$\text{Minimise} \quad \sum_{l=1}^{L} h_l^2 \alpha_l^2$$
$$\text{subject to} \quad \alpha_l \in [0, \alpha_{\mathsf{max}}], 1 \leq l \leq L,$$
$$\text{and} \quad \sum_{l=1}^{L} h_l \alpha_l = a \in [0, a_{\mathsf{max}}].$$

□

We guess a solution and verify via Karush-Kuhn-Tucker (KKT) conditions that the solution is optimal. (See also [3]).

The interval $[0, a_{\mathsf{max}}]$ can be broken into L closed intervals $[a_m, a_{m+1}], m = 0, 1, \ldots, L-1$, where $a_m = \alpha_{\mathsf{max}} \left(\sum_{l=1}^{m} h_l + (L-m) h_m\right)$ and $a_0 = 0$. The ordering of h_l's implies that $a_{m+1} \geq a_m$ so that each interval is nonempty.

Suppose that a in Problem 2 belongs to $[a_k, a_{k+1}]$. We guess that the optimal solution is
$$\alpha_l = \alpha_{\mathsf{max}}, \quad l = 1, \ldots, k, \quad (4)$$
$$\alpha_l = \frac{a - \alpha_{\mathsf{max}} \sum_{m=1}^{k} h_m}{(L-k) h_l}, \quad l = k+1, \ldots, L. \quad (5)$$
From the fact $a \in [a_k, a_{k+1}]$, it is easily verified that α meets all the constraints of Problem 2.

To see that this solves Problem 2, observe that the Lagrangian function for Problem 2 is
$$\mathcal{L} = \sum_{l=1}^{L} h_l^2 \alpha_l^2 + \sum_{l=1}^{L} \lambda_l (\alpha_l - \alpha_{\mathsf{max}}) - \sum_{l=1}^{L} \xi_l \alpha_l + \mu \left(\sum_{l=1}^{L} h_l \alpha_l - a \right),$$
where the Lagrange multipliers $\lambda_l \geq 0$, $\xi_l \geq 0$, and the KKT conditions are
$$\begin{aligned} \alpha_l &= \frac{\xi_l - \lambda_l - \mu h_l}{2 h_l^2}, \quad l = 1, \ldots, L, \\ \lambda_l (\alpha_l - \alpha_{\mathsf{max}}) &= 0, \quad l = 1, \ldots, L, \\ \xi_l \alpha_l &= 0, \quad l = 1, \ldots, L, \\ \sum_{l=1}^{L} h_l \alpha_l &= a. \end{aligned}$$

For the assignment of α above, consider the Lagrange multiplier assignments
$$\lambda_l = \frac{2 h_l}{L-k} \left(a - \alpha_{\mathsf{max}} \sum_{m=1}^{k} h_m \right) - 2 h_l^2 \alpha_{\mathsf{max}}, \quad l = 1, \ldots, k, \quad (6)$$
$$\begin{aligned} \lambda_l &= 0, \quad l = k+1, \ldots, L, \\ \xi_l &= 0, \quad l = 1, \ldots, L, \\ \mu &= -2 \left(\frac{a - \alpha_{\max} \sum_{m=1}^{k} h_m}{L-k} \right). \end{aligned}$$
These assignments will satisfy all KKT conditions if λ_l in (6) are non-negative. This holds because $a \in [a_k, a_{k+1}]$ implies that
$$\begin{aligned} \lambda_l &\geq \frac{2 h_l}{L-k} \alpha_{\mathsf{max}} (L-k) h_k - 2 h_l^2 \alpha_{\mathsf{max}} \\ &= 2 h_l (h_k - h_l) \alpha_{\mathsf{max}} \\ &\geq 0, \end{aligned}$$

where the last inequality follows because of the ordering of h_l's. The optimal solution to Problem 2 is thus given by (4) and (5), and the optimal value is

$$V(a) = \alpha_{\mathsf{max}}^2 \sum_{l=1}^{k} h_l^2 + \frac{\left(a - \alpha_{\mathsf{max}} \sum_{l=1}^{k} h_l\right)^2}{L-k}.$$

Observe that $V(a)$ is a continuous function of a for $a \in [0, a_{\mathsf{max}}]$. Furthermore, it is piecewise parabolic. This observation will be used in the sequel.

Our solution to Problem 2 shows that Problem 1 can be solved by solving:

Problem 3:

$$\text{Minimise} \quad f(a) = \frac{\sigma_{\mathsf{obs}}^2 V(a) + \sigma_{\mathsf{MAC}}^2}{a^2}$$

$$\text{subject to} \quad 0 \le a \le \alpha_{\mathsf{max}} \sum_{l=1}^{L} h_l.$$

□

Let us look at the objective function[1] when $a \in [a_m, a_{m+1}]$. By equating the derivative of this objective function to 0, we get that the minimum value[2] is attained at a_m^* given by

$$a_m^* = \alpha_{\mathsf{max}} \sum_{l=1}^{m} h_l + \frac{(L-m)\alpha_{\mathsf{max}} \sum_{l=1}^{m} h_l^2}{\sum_{l=1}^{m} h_l} + \frac{\sigma_{\mathsf{MAC}}^2 (L-m)}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}} \sum_{l=1}^{m} h_l}.$$

Now, the condition $a_m^* \in [a_m, a_{m+1}]$ is equivalent to

$$h_m \sum_{l=1}^{m} h_l \le \sum_{l=1}^{m} h_l^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2} \le h_{m+1} \sum_{l=1}^{m} h_l, \qquad (7)$$

where the lower bound is 0 if $m = 0$. If $a_m^* < a_m$, the optimum value is at a_m, and if $a_m^* > a_{m+1}$, the optimum value is at a_{m+1}. Otherwise, a_m^* lies in the interval and is the point of minimum for the objective function. Thus for each interval we have one candidate minimum. The overall minimum is the minimum of these finite number of candidates. We next make the following observations that identify further structure in the problem and provide an explicit solution.

1) For interval $[a_0, a_1]$, the optimal point is always a_1. This is because when $a \in [a_0, a_1]$, we have

$$\alpha_l = \frac{a}{L h_l}, \forall l, \quad \text{and}$$

$$f(a) = \frac{\sigma_{\mathsf{obs}}^2}{L} + \frac{\sigma_{\mathsf{MAC}}^2}{a^2}.$$

Clearly f is minimized at the upper limit a_1. As this is the lower limit of the next interval $[a_1, a_2]$, we may discard the case $a \in [a_0, a_1]$.

2) For interval $[a_1, a_2]$, $a_1^* > a_1$, i.e., optimal point for the objective function corresponding to the interval is either in the interval or to the right of the interval. This is because we trivially have

$$h_1^2 < h_1^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2},$$

and therefore (7) implies that a_1^* can never fall to the left of the interval.

3) If $a_m^* > a_{m+1}$, then $a_{m+1}^* > a_{m+1}$, i.e., if the optimal point for interval $[a_m, a_{m+1}]$ lies to the right of the interval, then the optimal point for interval $[a_{m+1}, a_{m+2}]$ either lies in or to the right of the interval. Indeed, if

$$\sum_{l=1}^{m} h_l^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2} > h_{m+1} \sum_{l=1}^{m} h_l,$$

we then have

$$\begin{aligned} \sum_{l=1}^{m+1} h_l^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2} &> h_{m+1} \sum_{l=1}^{m} h_l + h_{m+1}^2 \\ &= h_{m+1} \sum_{l=1}^{m+1} h_l. \end{aligned}$$

4) If $a_m^* \le a_{m+1}$, then $a_n^* \le a_n, n > m$, i.e., if the optimal point for interval $[a_m, a_{m+1}]$ lies within or to the left of the interval, then the optimal points for intervals $[a_n, a_{n+1}], n > m$, lie to their left. To see this, observe that if

$$\sum_{l=1}^{m} h_l^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2} \le h_{m+1} \sum_{l=1}^{m} h_l,$$

then

$$\begin{aligned} \sum_{l=1}^{m+1} h_l^2 + \frac{\sigma_{\mathsf{MAC}}^2}{\sigma_{\mathsf{obs}}^2 \alpha_{\mathsf{max}}^2} &\le h_{m+1} \sum_{l=1}^{m} h_l + h_{m+1}^2 \\ &= h_{m+1} \sum_{l=1}^{m+1} h_l. \end{aligned}$$

Since $a_n \le a_{n+1}$, the statement follows by induction.

The third observation above indicates that we may search sequentially, i.e., in the increasing order of m, for the index k that satisfies $a_k^* \in [a_k, a_{k+1}]$, or equivalently, for the smallest index k that satisfies (1). $V(a)$ being continuous, the third observation above shows that f decreases until we reach this point. The fourth observation and the fact that $V(a)$ is continuous shows that f increases beyond a_k^*. If there is no such k, the third observation indicates that we should pick the largest value of a, i.e., a_{max}, and thus $k = L$.

[1]Note that f may not be convex.

[2]That the stationary point is a minimum is deduced by recognising that V is a convex parabola over the interval and therefore $g(b) := f(1/b)$ is another convex parabola over a new interval whose end points are the inverses of the original interval's end points.

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References

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