Applied Probability Trust (1 October 2009)

OPTIMAL MULTICOMMODITY FLOW THROUGH THE COMPLETE GRAPH WITH RANDOM EDGE-CAPACITIES

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Abstract

We consider a multicommodity flow problem on a complete graph with the edges having random i.i.d capacities. We show that as the number of nodes tends to infinity, the maximum utility, given by the average of a concave function of each commodity flow, has an almost sure limit. Further, the asymptotically optimal flow uses only direct and two-hop paths, and can be obtained in a distributed manner.

Keywords: Complete graph; fairness; multicommodity flow; routing; utility

2000 Mathematics Subject Classification: Primary 05C21

Secondary 05C80; 90C25

1. Introduction

Flow maximisation on a graph is a central problem in graph theory and optimisation. The single source and single sink flow problem has been studied extensively and several algorithms have been developed for obtaining the maximum flow. An important case of flow problems that can be used to model realistic networks is the multicommodity flow, in which there is simultaneous flow between each source-destination pair. In this paper, we consider an edge-capacitated undirected graph. We associate a utility to the flow between each source-destination vertex pair, and seek to optimise the average utility of the flows. We first describe the problem and its solution. Towards the end of this section we indicate how our problem arises in practice.

For a given source v and destination w, the associated flow between them is con-

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served at all vertices except v and w. Writing $\varphi_{vw}(e)$ as the absolute value of this flow on an edge e, the volume of this vw flow is given by

$$f_{vw} = \sum_{e:e \text{ incident on } v} \varphi_{vw}(e) = \sum_{e:e \text{ incident on } w} \varphi_{vw}(e).$$

Assume that each pair of vertices of the graph forms a source-destination pair with the source-destination labelling chosen arbitrarily. Then given capacities $C(e)$ for edges e, we say that the flow profile $\{f_{vw}\}_{v,w}$ obtained via $\{\varphi_{vw}(e)\}_{e,v,w}$ is *feasible* if

$$\sum_{\{v,w\}} \varphi_{vw}(e) \leq C(e) \quad \forall e.$$

We consider the complete n-vertex graph G_n with random edge-capacities, and quantify the behaviour of the average utility as $n \to \infty$. Such a model was studied by Aldous et al. in [1] under the setting of uniform multicommodity flow, where all flows are required to be of the same volume. We interchangeably use the notation $C(e)$ or C_{vw} for the capacity of an edge e incident on vertices v and w. We assume that the capacities C_{vw} are independent and identically distributed (i.i.d.) copies of a reference random variable C that takes values in a set $\mathscr{C} \subseteq \mathbf{R}^+$ and satisfies $0 < \mathbb{E}[C] < \infty$.

For a given feasible flow profile $\{f_{vw}\}_{v,w}$, we define the *utility* of the flow profile to be

$$U_n = \frac{1}{a_n} \sum_{\{v,w\}} \zeta(f_{vw})$$

where $\zeta : \mathbf{R}^+ \to \mathbf{R} \bigcup \{-\infty\}$ is a strictly concave, increasing utility function with a continuous first derivative, $\zeta(x) > -\infty$ if $x > 0$, and $a_n = \binom{n}{2}$ is the number of edges. The maximum utility is denoted by

$$\rho_n = \left\{ U_n \,\middle|\, \{f_{vw}\}_{v,w} \text{ feasible} \right\}.$$

Examples of such utility functions are the so-called α-fair utility functions [10]

$$\zeta_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \alpha \in [0,\infty), \alpha \neq 1, \\ \log x & \alpha = 1. \end{cases}$$

As $\alpha \to \infty$, we have

$$\lim_{\alpha \to \infty} U_n^{1/(1-\alpha)} = \min_{\{v,w\}} f_{vw},$$

and the problem reduces to that of uniform multicommodity flow of Aldous et al. [1]. The solution to this problem may be thought of as a max-min fair solution. Aldous et al. [1] show that

- ρ_n for the uniform flow case converges in probability to a constant that depends on the distribution of C.
- Each flow may be routed through only direct and two-hop paths.

Their proof technique does not appear to be amenable to a distributed implementation.

Instead of choosing arbitrary source-destination labellings for a given pair of vertices as in our model, Aldous et al. [1] consider every ordered pair as a source-destination pair. We can frame our problem in that context by interpreting f_{vw} as the volume of flow from v to w and f_{wv} as the volume of flow from w to v and using ordered pairs (v, w) in the definition of U_n. However, concavity of ζ implies that flows in either direction should be equal for optimality. We therefore do not distinguish between f_{vw} and f_{wv} and let f_{vw} denote the net flow between v and w with one of them arbitrarily taken as source and the other as destination.

Our main results are the following.

- ρ_n converges with probability 1 to a constant

  $$\mathbb{E}\left[\zeta(h(C))\right]$$

  where h is the piecewise linear function truncated at a and saturated at b:

  $$h(x) = \begin{cases} a & x \leq a, \\ x & a \leq x \leq b, \\ b & x \geq b. \end{cases}$$

  The constants a and b depend on the distribution of C.
- Each flow f_{vw} in the asymptotically optimal flow profile is given by $h(C_{vw})$, a function of the capacity of the direct edge alone.
- Each flow requires only direct paths and two-hop paths.
- Our solution to find the flow profile is amenable to a distributed implementation.

Multicommodity flow problems were introduced as different from the single source, single sink problem in [9], and an algorithm for obtaining max-min fair optimal flow was described. Such problems arise in computer communication and wireless networks. Algorithms for solving multicommodity flow problems with fixed demands and capacities were described in [6, 8]. Flows over networks with random-edge capacities, where the capacities form a stochastic process with time as a parameter, were studied in the monograph [3] and references therein. One objective that was considered was to maximise the sum of concave utilities [10] arising from flow values. See [7] for a nonrandom version where there is only one route per flow. Georgiadis et al. [3] consider several generalisations with multiple routes, dynamic routing, random arrivals, and queues. The problem considered by Aldous et al. [1] and ours in this paper may be regarded as an asymptotic version of the simplest of these problems, with no queues and no time-variations, but with network size growing to infinity and one commodity per pair of vertices. This tractable asymptotic version may provide useful bounds for other intractable problems. Related problems along these asymptotic lines are those of flows between the top and bottom surfaces of a lattice with random edge-capacities [2, 4, 11].

The rest of the paper is organised as follows. In section 2, we solve the problem when ζ is linear. Section 3 provides conditions that ensure achievability of a utility when ζ is strictly concave, and describes a distributed method to obtain the corresponding feasible flow. We optimise the lower bound subject to these conditions in section 4, and prove that this is, in fact, optimal in section 5. Some final remarks in section 6 conclude the paper.

2. Linear utility

In this section, we consider the linear utility function $\zeta(x) = x$. Note that this is not strictly concave. However, it turns out that the optimal flow profile for this problem is also optimal for some strictly concave ζ's as will be highlighted later.

Theorem 1. *If $\zeta(x) \equiv x$, then $\rho_n \to \mathbb{E}[C]$ as $n \to \infty$ with probability 1.*

Proof. Let $f_{vw} = C_{vw}\ \forall \{v, w\}$. This flow profile is clearly feasible because each flow uses only the direct link to its capacity. For this allocation,

$$U_n = \frac{1}{a_n} \sum_{\{v,w\}} C_{vw}. \tag{1}$$

Next, let $\{\varphi_{vw}(e)\}_{e,v,w}$ form a feasible flow The capacity constraints are

$$\sum_{\{v,w\}} \varphi_{vw}(e) \leq C(e) \quad \forall e.$$

Summing over all edges e and interchanging the summations, we get

$$\sum_{\{v,w\}} \sum_{e} \varphi_{vw}(e) \leq \sum_{e} C(e). \tag{2}$$

We also have

$$f_{vw} = \sum_{e:e \text{ incident on } v} \varphi_{vw}(e) \leq \sum_{e} \varphi_{vw}(e) \quad \forall \{v,w\}.$$

Summing over all pairs $\{v,w\}$ and using (2), we get after dividing by a_n,

$$U_n = \frac{1}{a_n} \sum_{\{v,w\}} f_{vw} \leq \frac{1}{a_n} \sum_{e} C(e) \tag{3}$$

for any feasible flow. From (1), the upper bound in (3) is achievable, and hence

$$\rho_n = \frac{1}{a_n} \sum_{e} C(e),$$

which converges to $\mathbb{E}[C]$ as $n \to \infty$ with probability 1.

3. Achievability of flow

When ζ is linear, we saw in section 2 that the optimal flow is achieved by using only the direct link for each flow at its capacity. While this yields an efficient solution, the flow profile can be unfair. On the other hand, as proved in [1], the maximally fair asymptotically optimal flow profile is obtained using only direct and two-hop links. As such, it seems natural that the optimal flow in the case of a concave utility function, which enables operation between the two extremes, need not use more than two hops.

Now suppose that the flow volume f_{vw} depends only on the capacity of the direct link C_{vw} for all pairs $\{v,w\}$, i.e., $f_{vw} = h(C_{vw})$ for some $h : \mathscr{C} \to \mathbf{R}^+$. In this section, we obtain a sufficient condition (13) for such a flow to be feasible. Asymptotic optimality of such a flow is established in section 5.

For the uniform flow case, we set $h(C_{vw}) = \phi \; \forall \{v, w\}$, and remark that the condition (13) reduces to the necessary and sufficient condition for the feasible uniform flow as proved in [1]. Thus, the uniform multicommodity flow arises as a special case and the proof here serves as an alternative to the proof of achievability given by Aldous et al. in [1]. Our proof is elementary and is amenable to a distributed implementation.

3.1. Feasibility of certain integer flows

Here we show the achievability of certain integer flows with integer capacity constraints. This serves as the main tool to prove the main result of this paper. The proof is a modification of a procedure of Aldous et. al. [1].

Lemma 1. *Let C and F be random variables taking only nonnegative integer values. Let $M < \infty$ be an upper bound for both C and F. Let $\{(C_{vw}, F_{vw})\}_{v,w}$ be a set of i.i.d. pairs of random variables with each pair having the distribution of (C, F). If*

$$\mathbb{E}[(C - F)^+] - 2\mathbb{E}[(F - C)^+] > 0, \tag{4}$$

the flow on G_n obtained by setting $f_{vw} = F_{vw} \; \forall \{v, w\}$ is feasible for all but finitely many n, with probability 1.

Proof. Let C and F be such that (4) holds.

If $C_{vw} \geq F_{vw}$ for a given pair $\{v, w\}$, we use only the direct edge vw for the flow f_{vw}. Then, $C_{vw} - F_{vw}$ is the remaining capacity along edge vw. If $F_{vw} > C_{vw}$, we use the entire capacity C_{vw} of the direct edge for a part of f_{vw}. Then, $F_{vw} - C_{vw}$ is the remaining flow demand between v and w.

We decompose the original problem into M separate flow problems by constructing M graphs $P_1, P_2, \ldots,$
P_M, each with n vertices, as follows. For each vertex pair $\{v, w\}$, such that $F_{vw} > C_{vw}$, choose an $F_{vw} - C_{vw}$ size subset S_1 of $\{1, 2, \ldots, M\}$ uniformly and independently of other vertex pairs. For each $i \in S_1$, put a scarlet edge between v and w in graph P_i. Similarly, for each pair $\{v, w\}$, such that $C_{vw} \geq F_{vw}$, choose a $C_{vw} - F_{vw}$ size subset S_2 of $\{1, 2, \ldots, M\}$ uniformly and independently of other vertex pairs. For each $i \in S_2$, put a blue edge between v and w in graph P_i.

Now focus on one particular graph P_i. For a fixed vertex pair $\{v, w\}$, there is a scarlet edge between v and w in P_i with probability p_s given in (5), a blue edge with probability p_b given in (6), and no edge with the remaining probability $1 - p_s - p_b$. Also, this happens independently for all vertex pairs. As $\binom{M-1}{j-1}/\binom{M}{j} = j/M$ is the probability that a particular $i \in S_1$ given $|S_1| = j$, and analogously for $i \in S_2$ given $|S_2| = j$, we have

$$p_s = \sum_{j=1}^{M} \Pr\{F_{vw} - C_{vw} = j\} \frac{j}{M} = \frac{\mathbb{E}\left[(F-C)^+\right]}{M}, \tag{5}$$

$$p_b = \sum_{j=0}^{M} \Pr\{C_{vw} - F_{vw} = j\} \frac{j}{M} = \frac{\mathbb{E}\left[(C-F)^+\right]}{M}. \tag{6}$$

By the assumption in (4), we have $p_b > 2p_s$.

In the graph P_i, a scarlet edge between vertices v and w indicates a yet to be fulfilled unit demand for vw flow, and a blue edge between v and w indicates the availability of unit capacity along the edge vw. Thus, if in all $P_i, i = 1, 2, \ldots, M$, the demands along scarlet edges can be satisfied via the available capacities along blue edges, the flow $\{f_{vw} = F_{vw}\}_{v,w}$ can be achieved.

Such a problem is solved in [1] by using a packing result to form edge-disjoint triangles, each containing one scarlet and two blue edges, that cover all scarlet edges. Here we use an alternate method. The argument proceeds roughly as follows. A blue edge vw can potentially serve a vz flow for a vertex z if vz is a scarlet edge and wz is a blue edge. Similar is the case when wz is scarlet but vz is blue. By the nature of the colouring, the number of such vertices is a random variable having the binomial distribution with parameters $(n - 2, 2p_s p_b)$. The flow between two vertices having a scarlet edge between them can be served via a vertex connected with both by blue edges. The number of such vertices is a random variable having binomial distribution with parameters $(n-2, p_b^2)$. Dividing the flow across all such two-hop routes and using the concentration of the binomial distribution, we can get the required flows with high probability if $p_b^2 > 2p_s p_b$.

Formally, let vw be a blue edge. Define N_{vw} as the number of vertices $t \neq v, w$, such that t is connected to v, w by one scarlet and one blue edge.

Now, consider a scarlet edge vz. For all vertices $w \neq v, z$ connected to v, z by two blue edges, allocate a fractional flow of $1/\max(N_{vw}, N_{zw})$ through the 2-hop path

$v - w - z$. Do this for all scarlet edges. Then the flow through any blue edge is not greater than 1. The flow allocated for the scarlet vz is given by the random variable

$$\begin{aligned} R_{vz} &= \sum_{w \neq v,z} 1_{\{vw=\text{blue}\}} 1_{\{zw=\text{blue}\}} \times \frac{1}{\max(N_{vw}, N_{zw})} \\ &\geq J_{vz} \times \frac{1}{\max_{w \neq v,z} \{\max(N_{vw}, N_{zw})\}}, \end{aligned} \tag{7}$$

where $J_{vz} = \sum_{w \neq v,z} 1_{\{vw=\text{blue}\}} 1_{\{zw=\text{blue}\}}$ is a binomial random variable with parameters $(n-2, p_b^2)$. Note that for a fixed scarlet vz and fixed w with blue vw and zw, $N_{vw} - 1$ is a binomial $(n-3, 2p_s p_b)$ random variable, conditioned on z contributing 1 to the N_{vw} count.

Since $p_b > 2p_s$, we have $p_b^2 - 2p_s p_b > 0$. Choose ϵ such that $0 < \epsilon \leq (p_b^2 - 2p_s p_b)/2$. Then,

$$\frac{p_b^2 - \epsilon}{2p_s p_b + \epsilon} \geq 1. \tag{8}$$

From (7) and (8), the event

$$\begin{aligned} \{R_{vz} < 1\} \implies & \left\{J_{vz} \leq (n-2)(p_b^2 - \epsilon)\right\} \bigcup \\ & \left\{ \max_{w \neq v,z} \{\max(N_{vw}, N_{zw})\} \geq (n-2)(2p_s p_b + \epsilon) \right\}. \end{aligned} \tag{9}$$

By Bernstein's inequality [5, p.31],

$$\Pr\left\{J_{vz} \leq (n-2)(p_b^2 - \epsilon)\right\} \leq e^{-(n-2)\epsilon^2/4}. \tag{10}$$

Noting that

$$(n-2)(2p_s p_b + \epsilon) - 1 \geq (n-3)(2p_s p_b + \epsilon/2)$$

for all $n > 3 + 2/\epsilon$, we get

$$\Pr\left\{ \max_{w \neq v,z} \{\max(N_{vw}, N_{zw})\} \geq (n-2)(2p_s p_b + \epsilon) \right\} \leq 2(n-2)e^{-(n-3)\epsilon^2/16} \tag{11}$$

by the application of Bernstein's inequality and the union bound. Using (10) and (11) in (9), we get

$$\begin{aligned} \Pr\{R_{vz} < 1\} &\leq e^{-(n-2)\epsilon^2/4} + 2(n-2)e^{-(n-3)\epsilon^2/16} \\ &\leq 2ne^{-(n-3)\epsilon^2/16}. \end{aligned}$$

Since there are a maximum $n(n-1)/2 \le n^2/2$ scarlet edges, we have

$$\Pr\{R_{vz} < 1 \text{ for some scarlet edge } vz\} \le \frac{n^2}{2} 2ne^{-(n-3)\epsilon^2/16}.$$

Using the same procedure over all M graphs, and denoting by A_n the event that the flow profile $\{f_{vw} = F_{vw}\}_{v,w}$ on G_n is not feasible, i.e., there is some scarlet edge vz in one of the M graphs with $R_{vz} < 1$, we get

$$\Pr\{A_n\} \le Mn^3 e^{-(n-3)\epsilon^2/16}. \tag{12}$$

From (12), $\sum_{n=1}^{\infty} \Pr\{A_n\} < \infty$. This ensures, by Borel-Cantelli lemma [5, p.288], that the probability of A_n occurring infinitely often is 0. Hence, the flow $\{f_{vw} = F_{vw}\}_{v,w}$ on G_n is feasible for all but finitely many n, with probability 1.

3.2. Sufficient condition for a feasible flow

Lemma 2. *Let* $h : \mathscr{C} \to \mathbf{R}^+$ *be a function such that* $\inf_{x\in\mathscr{C}} h(x) > 0$ *and* $\sup_{x\in\mathscr{C}} h(x) < \infty$*. If*

$$\mathbb{E}\left[(C - h(C))^+\right] - 2\mathbb{E}\left[(h(C) - C)^+\right] \ge 0, \tag{13}$$

then $\liminf_{n\to\infty} \rho_n \ge \mathbb{E}\left[\zeta\left(h(C)\right)\right]$ *with probability 1.*

Proof. First observe that the expectation in (13) exists and is finite because $\mathbb{E}|h(C)-C| \le \mathbb{E}h(C)+\mathbb{E}C$, both of which exist and are finite. The expectation $\mathbb{E}[\zeta(h(C))]$ exists by Jensen's inequality and the assumption on ζ that $\zeta(x) > -\infty$ if $x > 0$.

Choose δ such that $0 < 2\delta < \inf_{x\in\mathscr{C}} h(x)$ and choose an integer k large enough that $k > 2/\delta$ and $k > \sup_{x\in\mathscr{C}} h(x)$. Define random variables

$$C^{(k)} = \frac{1}{k}\left\lfloor \min(kC, k^2)\right\rfloor \tag{14}$$

and

$$F^{(k)} = \frac{1}{k}\left\lfloor kh(C) - k\delta - 1\right\rfloor. \tag{15}$$

Observe that

$$0 \le C^{(k)} \le C$$

and

$$0 \le F^{(k)} \le \frac{1}{k}\left(kh(C) - k\delta - 1\right) \le k - \delta. \tag{16}$$

Moreover, $\left(kC_{vw}^{(k)}, kF_{vw}^{(k)}\right)$ are i.i.d. nonnegative integer quantities, so that we are in a position to apply Lemma 1 if we can verify (4) for $\left(C^{(k)}, F^{(k)}\right)$. To do this, we may write the expectation in (4) as an integral over $\{C \leq k\}$ and $\{C > k\}$, and use (14) and (16) to get

$$
\begin{aligned}
&\mathbb{E}\left[\left(C^{(k)} - F^{(k)}\right)^+ - 2\left(F^{(k)} - C^{(k)}\right)^+\right] \\
&\geq \frac{1}{k}\int_{c\leq k}\left[((kc-1) - (kh(c) - k\delta - 1))^+ - 2\left((kh(c) - k\delta - 1) - (kc-1)\right)^+\right] \mathrm{d}\mu(c) \\
&\qquad + \int_{c>k}\left[(k-k+\delta)^+ - 2(k-\delta-k)^+\right] \mathrm{d}\mu(c) \\
&= \int_{c\leq k}\left[(c - h(c) + \delta)^+ - 2\left(h(c) - \delta - c\right)^+\right] \mathrm{d}\mu(c) + \int_{c>k} \delta \,\mathrm{d}\mu(c) \\
&\overset{(a)}{\geq} \int_{c\leq k}\left[(c - h(c))^+ - 2\left(h(c) - c\right)^+ + \delta\right] \mathrm{d}\mu(c) + \int_{c>k} \delta \,\mathrm{d}\mu(c) \\
&= \int_{c\leq k}\left[(c - h(c))^+ - 2\left(h(c) - c\right)^+\right] \mathrm{d}\mu(c) + \delta, \qquad (17)
\end{aligned}
$$

where (a) follows because

$$(x+\delta)^+ - 2(x+\delta)^- \geq x^+ - 2x^- + \delta \quad \forall x \in \mathbf{R}.$$

By the dominated convergence theorem,

$$
\begin{aligned}
\lim_{k\to\infty}\int_{c\leq k}\left[(c - h(c))^+ - 2\left(h(c) - c\right)^+\right] \mathrm{d}\mu(c) &= \mathbb{E}[(C - h(C))^+ - 2(h(C) - C)^+] \\
&\geq 0.
\end{aligned}
$$

So choose k large enough that

$$\int_{c\leq k}\left[(c - h(c))^+ - 2\left(h(c) - c\right)^+\right] \mathrm{d}\mu(c) > -\delta.$$

Its substitution in (17) implies

$$\mathbb{E}\left[\left(C^{(k)} - F^{(k)}\right)^+ - 2\left(F^{(k)} - C^{(k)}\right)^+\right] > 0.$$

Hence by Lemma 1, the flow $\left\{f_{vw} = kF_{vw}^{(k)}\right\}_{v,w}$ is feasible for all but finitely many n, with probability 1 when we have integer capacities $\left\{kC_{vw}^{(k)}\right\}_{v,w}$. Scaling by $1/k$ and noting that $C^{(k)} \leq C$, the flow $\left\{f_{vw} = F_{vw}^{(k)}\right\}_{v,w}$ is feasible for all large enough k and for all but finitely many n, with probability 1. For this flow profile, the utility is

$$U_n^{(k)} = \frac{1}{a_n}\sum_{\{v,w\}} \zeta(F_{vw}^{(k)}).$$

Since $F^{(k)} \geq h(C) - 2\delta > 0$, as is easily verified, and ζ is an increasing function,

$$\begin{aligned} U_n^{(k)} &\geq \frac{1}{a_n} \sum_{\{v,w\}} \zeta(h(C_{vw}) - 2\delta) \\ &\geq \frac{1}{a_n} \sum_{\{v,w\}} \left[\zeta\left(h(C_{vw})\right) - 2\delta\zeta'\left(h(C_{vw}) - 2\delta\right)\right] \\ &\geq \frac{1}{a_n} \sum_{\{v,w\}} \left[\zeta\left(h(C_{vw})\right) - 2\delta\zeta'\left(\inf_{x\in\mathscr{C}} h(x) - 2\delta\right)\right]. \end{aligned}$$

The second inequality above follows from the strict concavity of ζ. Since

$$\delta\, \zeta'(\inf_{x\in\mathscr{C}} h(x) - 2\delta) \to 0$$

as $\delta \to 0$, we have

$$U_n^{(k)} \geq \frac{1}{a_n} \sum_{\{v,w\}} \zeta(h(C_{vw})) - \epsilon$$

for any $\epsilon > 0$. Noting that $\rho_n \geq U_n^{(k)}$, the event

$$B_\epsilon \triangleq \left\{\liminf_{n\to\infty} \rho_n \geq \mathbb{E}[\zeta(h(C))] - \epsilon\right\}$$

occurs with probability 1. Consequently, the event

$$B = \bigcap_{m=1}^{\infty} B_{1/m} = \left\{\liminf_{n\to\infty} \rho_n \geq \mathbb{E}\left[\zeta(h(C))\right]\right\}$$

also occurs with probability 1.

3.3. A distributed implementation

The proofs of Lemma 1 and Lemma 2 provide a randomised algorithm to obtain the feasible flow, which can be implemented in a distributed manner. The first step is to obtain an integer approximation as defined in (14) and (15). We may need to choose k large enough to get a utility sufficiently close to $\mathbb{E}\left[\zeta\left(h(C)\right)\right]$. Then, we use the algorithm in the proof of Lemma 1 to obtain a routing for this flow. Note that randomisation arises from the choice of the subsets that determine the edge colours in the M subgraphs. Here, we highlight the distributed nature of this algorithm.

The information available at an edge is assumed to be available also at its end-vertices. These include the capacity $C^{(k)}$, the flow requirement $F^{(k)}$, the presence or absence of the edge in each of the M graphs and their colours. Fix one of the M

graphs. Then each vertex will query each of its neighbours to obtain N_{vw} for each vertex w such that vw is blue. Then v exchanges the information on edge colours with each vertex z with vz scarlet to determine the vz flow on the path $v - w - z$ where vw and wz are both blue. This happens for each of the M graphs. Even though every node exchanges data with every other node, the graph is complete, the flow values and routes are determined based on locally available information.

4. Optimisation of the lower bound

Having found a sufficient condition (13) for feasible flow in Lemma 2, we optimise the utility over all such functions h. Recall that ζ is a strictly concave function. Consider the following functional optimisation problem:

$$\max_{h:\mathscr{C}\to\mathbf{R}^+} \mathbb{E}[\zeta(h(C))] \tag{18}$$

subject to $\mathbb{E}[(C - h(C))^+] - 2\mathbb{E}[(h(C) - C)^+] \geq 0$.

Let h^* be the optimising function. We will show that under the stated assumptions on ζ, h^* exists, so that the use of max in (18) is justified.

Let $\overline{\theta} = \lim_{x\downarrow 0} \zeta'(x)$ and $\underline{\theta} = \lim_{x\uparrow\infty} \zeta'(x)$. We have $0 \leq \underline{\theta} \leq \overline{\theta}$ and may assume $\underline{\theta} < \infty$ and $\overline{\theta} > 0$. Define

$$\psi(h) = \mathbb{E}\left[\zeta\left(h(C)\right)\right],$$

and

$$\xi(h) = \mathbb{E}\left[(C - h(C))^+\right] - 2\mathbb{E}\left[(h(C) - C)^+\right]. \tag{19}$$

Proposition 1. *If $\overline{\theta} \leq 2\underline{\theta}$, then $h^*(c) = c \; \forall c \in \mathscr{C}$.*

Proof. Choose $\lambda > 0$ such that

$$\overline{\theta}/2 \leq \lambda \leq \underline{\theta}. \tag{20}$$

Consider the function

$$\begin{aligned} w(h,\lambda) &= \psi(h) + \lambda\xi(h) \\ &= \mathbb{E}[\zeta(h(C)) + \lambda(C - h(C))^+ - 2\lambda(h(C) - C)^+]. \end{aligned} \tag{21}$$

We first maximise $w(h, \lambda)$ over all functions $h : \mathscr{C} \to \mathbf{R}^+$ for a fixed λ. Let the optimising function exist and be given by h_λ and suppose λ is such that $\xi(h_\lambda) = 0$.

Then, since $w(h_\lambda, \lambda) \geq w(h, \lambda)$, we have $\psi(h_\lambda) + \lambda\xi(h_\lambda) \geq \psi(h) + \lambda\xi(h)$. Thus, $\psi(h_\lambda) \geq \psi(h) + \lambda\xi(h) \geq \psi(h)$ over all functions h that satisfy $\xi(h) \geq 0$. Thus, h_λ is the optimising h^* for problem (18). We now prove the existence of such a λ and h_λ. We may write (21) as

$$w(h, \lambda) = \int_{\mathscr{C}} [\zeta(h(c)) + \lambda(c - h(c))^+ - 2\lambda(h(c) - c)^+] \mathrm{d}\mu(c).$$

Maximising $w(h, \lambda)$ is equivalent to maximising the integrand $\zeta(h(c)) + \lambda(c - h(c))^+ - 2\lambda(h(c) - c)^+$ pointwise for each $c \in [0, \infty)$. Thus, writing $h(c) = f$, we look to maximise

$$\zeta(f) + \lambda(c - f)^+ - 2\lambda(f - c)^+ \tag{22}$$

over $f \geq 0$ for a fixed c.

Define

$$g_1(f) = \zeta(f) - 2\lambda(f - c), \tag{23}$$

and

$$g_2(f) = \zeta(f) + \lambda(c - f). \tag{24}$$

The maximum value of (22) can be written in terms of g_1 and g_2 as

$$\max \left\{ \sup_{0 \leq f < c} g_2(f), \sup_{f > c} g_1(f), \zeta(c) \right\}. \tag{25}$$

The functions $g_1(f)$ and $g_2(f)$ are strictly concave functions in f. By the conditions on the slopes in the hypothesis and by (20), we have $g_1'(f) \leq 0$ and $g_2'(f) \geq 0$; so $g_1(f)$ is maximised at $f = 0$ and $g_2(f)$ is maximised at $f = \infty$. Because of concavity of $g_1(f)$ and $g_2(f)$, we have

$$g_1(0) \geq g_1(c) = \zeta(c) \geq g_1(f) \text{ for } c \leq f,$$
$$g_2(\infty) \geq g_2(c) = \zeta(c) \geq g_2(f) \text{ for } 0 \leq f \leq c,$$

The above equations imply that $\sup_{0 \leq f < c} g_2(f) \leq \zeta(c)$ and $\sup_{f > c} g_1(f) \leq \zeta(c)$. Thus, for each fixed c, the optimal value of (22) is $\zeta(c)$ and is achieved by setting $f = c$. Hence, the optimisation function $h_\lambda(c) = c \;\forall c \in \mathscr{C}$ and for any λ that satisfies (20). Further, $\xi(h_\lambda) = 0$, and hence, $h^*(c) \equiv c$ is the optimising function.

For the other case, $\overline{\theta} > 2\underline{\theta}$, we need the following definition. Define

$$\mathrm{SAT}(c, a, b) = \min(\max(a, c), b)$$

for given $a \leq b$ and

$$p_\lambda = \mathrm{SAT}\left(c, {\zeta'}^{-1}(2\lambda), {\zeta'}^{-1}(\lambda)\right)$$

for $\lambda \in \left[\underline{\theta}, \overline{\theta}/2\right]$.

Proposition 2. *If* $\overline{\theta} > 2\underline{\theta}$, *then*

$$h^*(c) = p_{\lambda^*}$$

where $\lambda^* \in \left[\underline{\theta}, \overline{\theta}/2\right] \bigcap [0, \infty)$ *is such that* $\xi(p_{\lambda^*}) = 0$.

Proof. Choose $\lambda > 0$ such that $\underline{\theta} \leq \lambda \leq \overline{\theta}/2$. We proceed as in the proof of Proposition 1 to maximise (22) for a fixed $c \in [0, \infty)$.

In this case, $g_1(f)$ and $g_2(f)$, defined in (23) and (24), have unique maxima $g_1(f_1)$ and $g_2(f_2)$ obtained at $f_1 = {\zeta'}^{-1}(2\lambda)$ and $f_2 = {\zeta'}^{-1}(\lambda)$ respectively. Observe that $f_1 < f_2$. Because of concavity of $g_1(f)$ and $g_2(f)$, we have the following inequalities under the specified cases on c.

$$g_1(f_1) \geq g_1(c) = \zeta(c) \geq g_1(f) \text{ for } f_1 \leq c \leq f, \tag{26}$$

$$g_2(f_2) \geq g_2(c) = \zeta(c) \geq g_2(f) \text{ for } f \leq c \leq f_2. \tag{27}$$

For $f \leq c \leq f_1$, and since $f_1 \leq f_2$, we have from (27), the condition

$$g_1(f_1) \geq \zeta(c) \geq g_2(f) \text{ for } f \leq c \leq f_1. \tag{28}$$

Analogously, for $f_2 \leq c \leq f$, and since $f_1 \leq f_2$, we have from (26), the condition

$$g_2(f_2) \geq \zeta(c) \geq g_1(f) \text{ for } f_2 \leq c \leq f. \tag{29}$$

For $c < f_1$, $f^* = f_1$ maximises (25) because of (28). Similarly, for $c > f_2$, $f^* = f_2$ maximises (25) because of (29). For $f_1 \leq c \leq f_2$, $f^* = c$ maximises (25) because of (26) and (27). Hence, $h_\lambda = p_\lambda$ maximises $w(h, \lambda)$.

We next check that there exists a $\lambda^* \in \left[\underline{\theta}, \overline{\theta}/2\right]$ with $\xi(h_{\lambda^*}) = 0$. Note that

$$\xi(h_\lambda) = \int_{\mathscr{C}} \left[\left(c - {\zeta'}^{-1}(\lambda)\right) 1_{\left\{c > {\zeta'}^{-1}(\lambda)\right\}} - 2\left({\zeta'}^{-1}(2\lambda) - c\right) 1_{\left\{c < {\zeta'}^{-1}(2\lambda)\right\}}\right] \mathrm{d}\mu(c). \tag{30}$$

$\xi(h_\lambda)$ is a continuous function in λ. Also, $\xi(h_{\underline{\theta}}) \leq 0$ and $\xi(h_{\overline{\theta}/2}) \geq 0$. Hence, there exists a $\lambda^* \in \left[\underline{\theta}, \overline{\theta}/2\right]$ such that $\xi(h_{\lambda^*}) = 0$. Note that if $\overline{\theta} = \infty$, then $\lim_{\lambda \uparrow \infty} \xi(h_\lambda) = \mathbb{E}[C] > 0$. Thus, the λ^* that solves $\xi(h_\lambda) = 0$ is finite.

Using the observations of the above propositions with Lemma 2, we have the following result.

Theorem 2. *Let U^* be the optimal solution to (18). Then,* $\liminf_{n\to\infty} \rho_n \geq U^*$ *with probability 1.*

Proof. Suppose $\overline{\theta} \leq 2\underline{\theta}$. Then by Proposition 1, h^* solving (18) is $h^*(c) = c$. In this case, $U^* = \mathbb{E}[\zeta(C)]$ can be achieved via the flow profile $\{f_{vw} = C_{vw}\}_{v,w}$, which is shown to be feasible in Theorem 1.

Now, suppose $\overline{\theta} > 2\underline{\theta}$. We saw in Proposition 2 that there exists a

$$\lambda^* \in \left[\underline{\theta}, \overline{\theta}/2\right] \bigcap [0, \infty)$$

with $\xi(h_{\lambda^*}) = 0$, and

$$h^*(c) = h_{\lambda^*}(c) = \mathrm{SAT}\left(c, {\zeta'}^{-1}(2\lambda^*), {\zeta'}^{-1}(\lambda^*)\right).$$

Suppose $\lambda^* \in \left(\underline{\theta}, \overline{\theta}/2\right)$. Then, the optimising function $h^*(c)$ is bounded below by ${\zeta'}^{-1}(2\lambda^*) > 0$ and is bounded above by ${\zeta'}^{-1}(\lambda^*) < \infty$. Then by Lemma 2, the corresponding U^* is achievable.

If $\lambda^* = \underline{\theta}$, then ${\zeta'}^{-1}(\lambda^*) = \infty$. Then using (30), we have

$$\xi(h_{\lambda^*}) = 0 \implies \Pr\left\{C < {\zeta'}^{-1}(2\lambda^*)\right\} = 0.$$

In this case, $h^*(c) = c$ over a set with probability 1. Similarly, if $\lambda^* = \overline{\theta}/2$, then ${\zeta'}^{-1}(2\lambda^*) = 0$, and therefore by (30), we have

$$\xi(h_{\lambda^*}) = 0 \implies \Pr\left\{C > {\zeta'}^{-1}(\lambda^*)\right\} = 0.$$

In this case also, $h^*(c) = c$ over a set with probability 1. Hence, in the above two cases, U^* is achievable via the flow profile $\{f_{vw} = h^*(C_{vw})\}_{v,w}$, which is feasible with probability 1.

5. Converse

In this section, we prove the converse of Theorem 2.

Theorem 3. *Let U^* be the optimal solution to (18). Then,* $\limsup_{n\to\infty} \rho_n \le U^*$ *with probability 1.*

Proof. First, we proceed as in Aldous et al. [1], to get a necessary condition for any flow (see (33) below).

Consider an arbitrary capacity realisation $\{C_{vw}\}_{v,w}$. For any pair $\{v, w\}$, we have

$$f_{vw} = \sum_{e:e \text{ incident on } v} \varphi_{vw}(e),$$

and therefore

$$\sum_e \varphi_{vw}(e) \ge f_{vw}. \tag{31}$$

For pairs $\{v, w\}$ such that $f_{vw} > C_{vw}$, since at least $f_{vw} - C_{vw}$ flow has to be carried by a path of length two or more, we have the stronger condition

$$\sum_e \varphi_{vw}(e) \ge C_{vw} + 2(f_{vw} - C_{vw}). \tag{32}$$

Combining (31) and (32), we have

$$\sum_e \varphi_{vw}(e) \ge \min\{f_{vw}, C_{vw}\} + 2(f_{vw} - C_{vw})^+.$$

Summing over all $\{v, w\}$ pairs and using (2), we get

$$\sum_e C(e) \ge \sum_{\{v,w\}} \left(\min\{f_{vw}, C_{vw}\} + 2(f_{vw} - C_{vw})^+\right).$$

Division by a_n and rearrangement yields

$$\frac{1}{a_n} \sum_{\{v,w\}} \left((C_{vw} - f_{vw})^+ - 2(f_{vw} - C_{vw})^+\right) \ge 0, \tag{33}$$

a necessary condition for any flow $\{f_{vw}\}_{v,w}$ to be feasible. This was obtained by Aldous et al. [1] in the context of uniform multicommodity flow with $f_{vw} = \phi \ \forall \{v, w\}$. But we see that (33) holds for any feasible flow.

It follows that ρ_n is always less than or equal to the solution to the following optimisation problem:

$$\max \frac{1}{a_n} \sum_{\{v,w\}} \zeta(f_{vw})$$

subject to

$$\frac{1}{a_n}\sum_{\{v,w\}}\left((C_{vw}-f_{vw})^+-2(f_{vw}-C_{vw})^+\right)\geq 0.$$

Let $\hat{U}$ be the optimal solution to this problem. Let $f=\{f_{vw}\}_{v,w}$, and define

$$\hat{\psi}(f)=\frac{1}{a_n}\sum_{\{v,w\}}\zeta(f_{vw}), \tag{34}$$

$$\hat{\xi}(f)=\frac{1}{a_n}\sum_{\{v,w\}}\left((C_{vw}-f_{vw})^+-2(f_{vw}-C_{vw})^+\right). \tag{35}$$

For $\lambda>0$, independent of the realisation, define $\hat{w}(f,\lambda)=\hat{\psi}(f)+\lambda\hat{\xi}(f)$. We first optimise $\hat{w}(f,\lambda)$ for a fixed $\lambda>0$.

Using (34) and (35), we get

$$\hat{w}(f,\lambda)=\frac{1}{a_n}\sum_{\{v,w\}}\left(\zeta(f_{vw})+\lambda(C_{vw}-f_{vw})^+-2\lambda(f_{vw}-C_{vw})^+\right).$$

The maximisation of $\hat{w}(f,\lambda)$ is separable in $\{v,w\}$, and therefore, we optimise the summand for each $\{v,w\}$ by choosing an appropriate f_{vw}.

Comparing with the optimisation of $w(h,\lambda)$ in section 4, the optimising flow f_λ is of the same form, i.e., $f_{\lambda,vw}=h_\lambda(C_{vw})$.

If $\overline{\theta}\leq 2\underline{\theta}$, then for λ such that $\overline{\theta}/2\leq\lambda\leq\underline{\theta}$, f_λ is given by $f_{\lambda,vw}=C_{vw}\ \forall\{v,w\}$, as obtained in the proof of Proposition 1. In this case, $\hat{\xi}(f_\lambda)=0$ and

$$\hat{\psi}(f_\lambda)=\frac{1}{a_n}\sum_{\{v,w\}}\zeta(C_{vw}).$$

Thus,

$$\limsup_{n\to\infty}\rho_n\leq\limsup_{n\to\infty}\frac{1}{a_n}\sum_{\{v,w\}}\zeta(C_{vw}).$$

The right-hand side is almost surely $\mathbb{E}[\zeta(C)]$, which is equal to U^* in this case, and so $\limsup_{n\to\infty}\rho_n\leq U^*$ with probability 1.

If $\overline{\theta}>2\underline{\theta}$, choose $\lambda=\lambda^*\in\left[\underline{\theta},\overline{\theta}/2\right]\bigcap[0,\infty)$ so that ξ as defined in (19) satisfies $\xi(h_{\lambda^*})=0$ (Note the distinction between ξ and $\hat{\xi}$). As discussed in Proposition 2, such a λ^* exists and is independent of the realisation with which we are now working. With this λ^*, $f_{\lambda^*,vw}=h_{\lambda^*}(C_{vw})=h^*(C_{vw})$.

Now,

$$\hat{w}(f_{\lambda^*},\lambda^*)\geq\hat{w}(f,\lambda^*),$$

which implies

$$\hat{\psi}(f_{\lambda^*}) + \lambda^* \hat{\xi}(f_{\lambda^*}) \geq \hat{\psi}(f) + \lambda^* \hat{\xi}(f),$$

and therefore

$$\hat{\psi}(f) \leq \hat{\psi}(f_{\lambda^*}) - \lambda^* \left(\hat{\xi}(f) - \hat{\xi}(f_{\lambda^*}) \right).$$

Hence, for all flow profiles f that satisfy $\hat{\xi}(f) \geq 0$, we have

$$\hat{\psi}(f) \leq \hat{\psi}(f_{\lambda^*}) + \lambda^* \hat{\xi}(f_{\lambda^*}),$$

which implies that

$$\rho_n \leq \hat{U} \leq \hat{\psi}(f_{\lambda^*}) + \lambda^* \hat{\xi}(f_{\lambda^*}). \tag{36}$$

Note that

$$\hat{\psi}(f_{\lambda^*}) = \frac{1}{a_n} \sum_{\{v,w\}} \zeta\left(h^*(C_{vw})\right),$$

which converges to $\mathbb{E}\left[\zeta\left(h^*(C)\right)\right] = U^*$ with probability 1. Also,

$$\hat{\xi}(f_{\lambda^*}) = \frac{1}{a_n} \sum_{\{v,w\}} \left((C_{vw} - h^*(C_{vw}))^+ - 2(h^*(C_{vw}) - C_{vw})^+ \right),$$

which converges to

$$\mathbb{E}\left[(C - h^*(C))^+ - 2\left(h^*(C) - C\right)^+ \right] = \xi(h^*) = 0,$$

with probability 1. Thus, taking lim sup in (36), we have $\limsup_{n\to\infty} \rho_n \leq U^*$, and the proof is complete.

6. Conclusion

We studied the asymptotic behaviour of optimal flows on the complete graph. The optimal net utility converges with probability 1 to a value that depends on the distribution of C. Interestingly, the volume of each flow depends only on the capacity of the corresponding direct link via a simple function. More precisely, we have shown the following.

1. If the slope of the utility function ζ at the origin is less than twice the slope at infinity, i.e., $\overline{\theta} \leq 2\underline{\theta}$, then $\lim_{n\to\infty} \rho_n = \mathbb{E}\left[\zeta(C)\right]$ with probability 1, and it is optimal to route each flow entirely via the direct link.

2. If $\overline{\theta} > 2\underline{\theta}$, then $\lim_{n\to\infty} \rho_n = \mathbb{E}\left[\zeta\left(h^*(C)\right)\right]$ with probability 1, where

$$h^*(c) = \mathrm{SAT}\left(c, {\zeta'}^{-1}(2\lambda^*), {\zeta'}^{-1}(\lambda^*)\right)$$

and λ^* solves

$$\mathbb{E}\left[(C - h^*(C))^+\right] - 2\mathbb{E}\left[(h^*(C) - C)^+\right] = 0.$$

The flows for each pair $\{v, w\}$ is $h^*(C_{vw})$ and is routed through only direct and two-hop routes. The resultant flow profile can be obtained through a simple distributed algorithm that requires information sharing only among links that share a vertex.

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