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Local Weak Convergence Based Analysis of a New Graph Model

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EXTENDED ABSTRACT

Different random graph models have been proposed as an attempt to model individuals' behavior. Each of these models proposes a unique way to construct a random graph that covers some properties of the real-world networks. In a recent work[4], the proposed model tries to capture the self-optimizing behavior of the individuals in which the links are made based on the cost/benefit of the connection. In this paper, we analyze the asymptotics of this graph model. We prove the model locally weakly converges [1] to a rooted tree associated with a branching process which we named **E**rlang **W**eighted **T**ree(**EWT**) and analyze the main properties of the EWT.

The graph construction starts with a complete graph $K_n = ([n], E_n)$ and a random function W_n that assigns positive integers to the nodes and non-negative real values to the edges of K_n, independently. The value of the node i is distributed as $P(\cdot)$ and indicates the number of neighbors that s/he wants to connect to. The value assigned to each edge is an exponentially distributed random variable with parameter $1/n$ that represents the cost of the edge. Thereafter, each node i selects the $W(i)$ lowest cost incident edges and declares them to be preferred edges. The random graph $G_n = ([n], \widetilde{E}_n)$ is constructed by keeping the edges of E_n that is preferred by both end nodes.

As the first step to analyze the asymptotics of the random graph model, we prove $\mathbb{E}(U_{G_n})$, where U_{G_n} is the uniform measure generated by G_n over the space of rooted graphs, converges weakly to $Er(P)$, the unimodular probability measure associated with EWT. Next, we derive the main properties of the EWT such as the probability of extinction, emergence of phase transition, growth/extinction rate, etc.

The branching process EWT is defined as follows; let $\mathbb{N}^f = \cup_{k \geq 0} \mathbb{N}^k$, where $\mathbb{N}^0 := \phi$ as a convention. Each $\boldsymbol{i} \in \mathbb{N}^f$ is associated with three types of random variables: $1) n_{\boldsymbol{i}}$ which is the potential number of neighbors of node $\boldsymbol{i}$, $2) v_{\boldsymbol{i}}$ which is the mark of node i, and, $3) \{\zeta_{(\boldsymbol{i},j)}\}_{j=1}^{n_{\boldsymbol{i}}}$ which represents the marks over the potential links $\{\boldsymbol{i}, (\boldsymbol{i}, j)\}$ for $j \in \{1, 2, \ldots, n_{\boldsymbol{i}}\}$. The probability distribution of n_ϕ is given by $P \in \mathcal{P}(\mathbb{Z}_+)$—which is assumed to have positive finite mean and $P(0) = 0$—and the probability distribution of $n_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}^f \setminus \mathbb{N}^0$ is given by shifted distribution $\hat{P} \in \mathcal{P}(\mathbb{Z}_+)$, i.e., $\hat{P}(k-1) = P(k)$ for all $k \geq 1$. Conditioned on $n_{\boldsymbol{i}}$, $v_{\boldsymbol{i}}$ is distributed as $Erlang(n_{\boldsymbol{i}} + 1, 1)$. Conditioned on $n_{\boldsymbol{i}}$ and $v_{\boldsymbol{i}}$, $\{\zeta_{(\boldsymbol{i},j)}\}_{j=1}^{n_{\boldsymbol{i}}}$ are $n_{\boldsymbol{i}}$ independent and uniformly distributed random variables over the interval $[0, v_{\boldsymbol{i}}]$. Define a rooted tree $T = (V, E, \phi, w)$, rooted at ϕ, by putting an edge of length $\zeta_{(\boldsymbol{i},j)}$ between the nodes $\boldsymbol{i}$ and $(\boldsymbol{i}, j)$ if and only if $\zeta_{(\boldsymbol{i},j)} < v_{(\boldsymbol{i},j)}$, where the function $w(\cdot)$ is the mark function that operates over the nodes and the edges of the tree T,

$$w : V \to \mathbb{N} \times \mathbb{R}, \qquad w(\boldsymbol{i}) = (n_{\boldsymbol{i}}, v_{\boldsymbol{i}})$$
$$w : E \to \mathbb{R}, \qquad w(\{\boldsymbol{i}, (\boldsymbol{i}, j)\}) = \zeta_{(\boldsymbol{i},j)}$$

The random rooted tree T is called an Erlang Weighted Tree with *potential degree distribution* P and the probability distribution of $[T]$ in the space of rooted graphs is denoted by $Er(P)$. We follow the technique in [2] to prove the local weak convergence of the finite graph model to EWT.

Theorem 1 (Local Weak Convergence)**.** The finite graph model converges to EWT in local weak sense,

$$\mathbb{E}(U_{G_n}) \xrightarrow{w} Er(P)$$

The EWT has an interdependent structure. As a result, the degree distribution of different generations are different. However, conditioned on the type of a node, its degree is given as follows,

$$P\left(D_{\boldsymbol{i}} = d | n_{\boldsymbol{i}} = m, v_{\boldsymbol{i}} = x\right) = Bi\left(d; m, \int_0^x \frac{1}{x} \sum_{k=1}^{\infty} P(k) \bar{F}_k(y) dy\right)$$

where Bi stands for binomial distribution,

$$Bi(d; m, \lambda) = \binom{m}{d} \lambda^d (1-\lambda)^{m-d}$$

Next, we demonstrate the existence and uniqueness of the Perron-Frobenius eigenvalue and left/right eigenfunctions of the one-step growth operator. The approach we take is introduced by Harris in [3]; however, his results does not apply to our setting.

Let $M_l(x, m; A, k-1)$ denotes the expected number of nodes at depth l of type $(z, k-1)$ for $z \in A$, given the mark of the root node is (x, m) and let $m_l(x, m; z, k-1)$ denotes its density at $z \in \mathbb{R}$. Then the following theorem holds.

Theorem 2. Assume that the moment generating function of n_ϕ exists for some small enough $\theta > 0$. Let β_0 to be the largest positive zero of the function $L(\beta, 0)$ where,

$$L(\beta, x) = \sum_{i=0}^{\infty} G_i(x) \left(\frac{-1}{\beta}\right)^i$$

and the function $G_i(x)$ is defined recursively as follows,

$$G_i(x) = \int_x^\infty \int_{z=y}^\infty g_2(z) G_{i-1}(z) dz dy \qquad \forall i > 0$$
$$G_0(x) = 1$$

Define $f_0(x) := L(\beta_0, x)$. Then $0 < \beta_0 < E[n_\varnothing] - 1$ is the unique eigenvalue of M_1 in $\mathbb{R}$. The corresponding eigenfunctions are given as follows

$$\text{Right eigenfunction: } \mu(x, m) = \frac{m}{x} f_0(x)$$

$$\text{Left eigenfunction: } \nu(z, k-1) = P(k) \frac{e^{-z} z^{k-1}}{(k-1)!} f_0(z)$$

These eigenfunctions are the unique non-negative eigenfunctions and all the other eigenvalues of M_1 are smaller than β_0 in magnitude. Moreover, there exists $0 < \Delta < 1$ independent of x, m, z and k such that for all $x \in \mathbb{R}_+$, $y > 0$, $k \geq 1$ and $m \geq 0$,

$$m_l(x, m; z, k-1) = {\beta_0}^l \mu(x, m) \nu(z, k-1) \times \left(1 + \frac{O(\Delta^l)}{g_2(z) {f_0(z)}^2}\right)$$

Specifically, as l goes to ∞ the density $m_l(x, m; z, k-1)$ converges to ${\beta_0}^l \mu(x, m) \nu(z, k-1)$.

An immediate corollary is that the branching process extincts whenever $\beta_0 \leq 1$ and, it explodes with positive probability given $\beta_0 > 1$. However, to get the exact probability of extinction, more works need to be done.

Theorem 3. The probability of extinction conditioned on the mark of the root node to be (x, m) is given by,

$$P(\{extinction\}) = \sum_{m=1}^{\infty} P(m) \int_{x=0}^{\infty} \frac{e^{-x} x^m}{m!} (q(x))^m dx$$

where $q(\cdot)$ is the smallest fixed point of the operator $T : \Omega(\mathbb{R}_+, [0,1]) \to C^1(\mathbb{R}_+, [0,1])$, defined as,

$$T(f)(x) := \frac{1}{x} \sum_{k=1}^{\infty} P(k) \int_{y=0}^{x} \left(\int_{z=0}^{y} \frac{e^{-z} z^{k-1}}{(k-1)!} dz + \int_{z=y}^{\infty} \frac{e^{-z} z^{k-1}}{(k-1)!} f(z)^{k-1} dz \right) dy$$

Equivalently, the function $q(\cdot)$ is the point-wise limit of $T^l(\mathbf{0})(\cdot)$ as l goes to infinity, where the function $\mathbf{0}(\cdot)$ is the null function, i.e., $\mathbf{0}(x) = 0 \ \forall x$,

$$q(x) = \lim_{l \to \infty} T^l(\mathbf{0})(x)$$

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